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$N = 2$ AND $N = 4$ SUPERGRAVITIES AS COMPACTIFICATIONS FROM STRING THEORIES IN 10 DIMENSIONS.

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A mon père et mon grand-père qui auraient aimé être présents lors de l'achèvement de ce travail.

Contents

Chapter 1

Introduction

The early excitements about string theory came from its possible ability to reconcile General Relativity with Quantum Mechanics [1–3]. On the one hand, General Relativity explains the behaviour of gravity at macroscopic scales. Among its main predictions one can cite the deviation of light near a matter source, or the relativity and local dependence of time. On the other hand Quantum Mechanics brought new insights in the structure of matter at microscopic scales, and introduced the ideas of quantization and uncertainty. It explained the discrete spectrum of hydrogenoid atoms, or the black body radiation. Quantum Mechanics was extended to Quantum Field Theory to describe the interactions of particles when creation and annihilation processes take place. Three of the four fundamental forces, electromagnetism, weak and strong interactions, could be unified in the same Standard Model, a quantum field theory based on the gauge group $SU(3) \times SU(2) \times U(1)$. The Standard Model has given the most accurate results ever predicted by a physical theory. However, it suffers from important drawbacks. The gauge group, as well as the masses and coupling constants, are put "by hand". Their values are measured, and no internal principle can guide us to their origin. Moreover, the Standard Model is only defined in flat space-time, which means that it says nothing about the other fundamental force, gravity. General relativity describes with great accuracy the behaviour of gravity, but it does not include any possible quantum effects of this interaction. It has become clearer and clearer, for example regarding space-time singularities in black holes, that a quantum formulation of gravity has to be discovered. Besides this, the constant progress of Physics towards unification of all interactions is an indication that a formulation of the Standard Model where all forces are treated on the same footing may exist. This is where string theory enters the game.

The fundamental object, the string, has dimension 1. The corresponding action is a Quantum Field Theory on the world sheet swept by the strings as it moves in a D-dimensional space-time. It has invariance under Poincaré transformations, world sheet diffeomorphisms and Weyl symmetry. If one considers superstrings, the world sheet action is extended to have invariance under supersymmetry transformations. The mass of the string is determined in terms of its internal oscillation degrees of freedom according to

$$M^2 \sim \frac{1}{\alpha'}(N - A) \tag{1.1}$$

where A is a constant of zero-energy and N is the number of oscillations. For the open string, N is a general integer and $A = 1$, for the closed string N is even and $A = 2$. The ground state ($N = 0$) is a tachyon and is projected out by supersymmetry. The first excited level has zero-mass, and all other states have masses quantized in units of the Planck mass $\frac{1}{\sqrt{\alpha'}} \sim 10^{19}$ GeV. Of course, the massive states have no physical interest, and the particles of the Standard Model are to be found in the massless sector.

One should stress that, in string theory, the dimension of space-time is not set by hand, but is determined by consistency considerations. The Fock space contains states which, due to the negative signature of space-time, have negative norms. Only in the particular case of $D = 10$ are these states absent. The massless physical states thus correspond to representations of the

transverse group $SO(8)$. It turns out that one can distinguish 5 different string theories. Here we give their main characteristics and their bosonic massless spectra[1].

Type I It is composed of open and unoriented closed strings, has $N = 1$ supersymmetry, and contains the metric g_{MN}, an antisymmetric tensor B_{MN}, the dilaton ϕ, and 496 vectors A^a_M with gauge group $SO(32)$.

Heterotic $SO(32)$ or $E_8 \times E_8$ Heterotic theories are hybrids of closed strings and superstrings. They have $N = 1$ supersymmetry, and contain the metric g_{MN}, an antisymmetric tensor B_{MN}, the dilaton ϕ, and 496 vectors A^a_M with gauge group $SO(32)$ or $E_8 \times E_8$.

Type IIA It is made of closed strings, has $N = 2$ supersymmetry, contains the metric g_{MN}, an antisymmetric tensor B_{MN}, the dilaton ϕ, a vector A_M and a 3-form C_{MNP}.

Type IIB It is made of closed strings, has $N = 2$ supersymmetry, contains the metric g_{MN}, an antisymmetric tensor B_{MN}, the dilaton ϕ, a scalar l, a 2-form C_{MN} and a 4-form A_{MNPQ} with self-dual field strength.

At first the fact that several consistent string theories could exist seemed unappealing, because string theory was supposed to unify all forces in a single framework, and this lack of unicity was a serious drawback. But it was realized in the mid 90ies that several relations, the dualities, hold between these 5 theories. In this thesis we will concentrate on one of these dualities, mirror symmetry. Before describing it in more detail, we need to introduce the notion of phenomenology and compactification in string theory.

If string theory is to be viable, then, in some limit to define, it should be able to reduce to the Standard Model, which has given extremely accurate predictions for numerous experiments up to now. The mass scale of string theory is of Planck order, and the masses that today's experiments can probe are of order 1 Tev, so it is obvious that only a low-energy limit of string theory should give the Standard Model. As a first step, one restricts the spectrum to the massless particles, which are the only plausible candidates for the known particles. In a second step one needs to find a space-time action for these fields. To do so, one uses the constraints implied by Weyl symmetry of the string action. The massless fields obtained in the various spectra of the 5 string theories can serve as coupling functions in the world sheet action. However, not all field configurations preserve Weyl symmetry at the quantum level. There is a Weyl anomaly which lies in the trace of the energy-momentum tensor and is absent only if the β-functions governing the behaviour of each coupling vanish. This leads to a set of equations that take the form of equations of motion. These equations in turn can be obtained from a variational principle applied to a space-time action, the supergravity action. This new theory is the low-energy limit of the corresponding string theory.

A fundamental issue is then the search for consistent solutions to these equations of motion, called backgrounds or vacua. Finding a consistent background means giving vacuum expectation values to all fields of the theory in such a way that the equations of motion are satisfied. A very general supergravity describes the dynamics of a graviton, some p-forms, and a set of scalars (or 0-forms) in the bosonic sector, as well as their supersymmetry partners in the fermionic sector. If one's interest is restricted to massless fields without potential, flat Minkowski space-time with vanishing values for all fields (other than the metric) is always a solution. However one may be interested in more sophisticated backgrounds for which some fields acquire non vanishing values and/or the metric is no longer flat.

Finding all solutions is of course an extremely difficult problem; one starts instead from

[1] Supersymmetry insures that there is an equal number of bosonic and fermionic degrees of freedom. The fermionic spectra are generically composed of N gravitini and a number of spin 1/2 fermions such their degrees of freedom match the bosonic ones.

simplifying ansätze. In the Kaluza-Klein (KK) framework, the geometry of the D-dimensional space-time is that of a product of the space-time in $d < D$ dimensions times an internal manifold. Since in this thesis we want to consider theories in $d = 4$, the KK ansatz reads

$$\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{I}_6 \tag{1.2}$$

where $\mathcal{M}_{10}$ is the product of the 4-dimensional space-time $\mathcal{M}_4$ and an internal compact[2] manifold $\mathcal{I}_6$. On the fields, it amounts to a separation of space-time and internal variables. A 10-dimensional field $\hat{\phi}$ is expanded according to

$$\hat{\phi}(\hat{x}) = \phi_n(x) h_n(y) \tag{1.3}$$

where x and y are the space-time and internal coordinates, and h_n are harmonic functions on the internal manifold.

The geometry of the internal space has to be consistent with Einstein's equation

$$R_{MN} = F^{(i)}_{M..} F^{(i)..}_{N} \tag{1.4}$$

where R_{MN} is the Ricci tensor and $F^{(i)}_{M..}$ is an i-form field strength. If one assumes that the field strengths have no purely internal components, then the internal manifold has to be Ricci-flat.

Further information can be obtained by imposing a specific number of conserved supercharges in 4 dimensions. Such a condition highly constrains the internal manifold. Take for instance type II supergravities in 10 dimensions. These have invariance under the maximal number of supercharges, that is, 32. If one wants to obtain an $N = 2$ supergravity in 4 dimensions, the internal manifold has to preserve 1/4th of the supercharges. These manifolds are known as Calabi-Yau spaces. Calabi-Yau compactifications do not produce any potential for the moduli, the scalars that parameterize degenerate vacua. Consequently, the vacua correspond to arbitrary constant values of the moduli and are degenerate. Moreover, $N = 2$ supersymmetry remains completely unbroken.

However, for phenomenological reasons, it is interesting to look for vacua where $N = 2$ supersymmetry is partially or completely broken. To this end, one can relax the above constraints for the field strengths and allow for some purely internal components. For consistency with Bianchi Identities, the field strengths have to be expanded on the harmonic forms on the internal Calabi-Yau space, and the number of flux parameters is determined by the Betti number of the cohomology class which is expanded on. The main features of the introduction of such fluxes are the gauging of some isometries of the scalar manifold and the appearance of a scalar potential. The minimum of this potential is generally obtained for non-vanishing values of the scalars, and $N = 2$ supersymmetry is broken[3].

Under the relaxed constraints including fluxes, it is interesting to study the fate of mirror symmetry. Mirror symmetry is one of the dualities relating the 5 string theories in 10 dimensions. In the case of type II theories, it states that type IIA supergravity on a Calabi-Yau 3-fold is the same as type IIB on the mirror Calabi-Yau manifold, defined by the exchange of odd and even cohomologies. This symmetry still holds when Ramond-Ramond (R-R) fluxes are turned on. Recall that type II spectra are divided into two sectors. The Neveu-Schwarz (NS-NS) sector, common to both type II theories, contains the metric, the dilaton, and an antisymmetric tensor (or 2-form), while the R-R spectra contains only forms of different degrees, depending on the type of the supergravity. In type IIA, the R-R sector contains a 4-form and a 2-form field strengths. This means that the R-R fluxes lie in the even cohomologies. On the other side, in

[2]Recently it has been proposed that some non compact spaces may play the role of internal manifold, leading to large extra dimensions of millimeter size. The mass scale of the string theory is no longer the Planck mass, but a mass of order 1 Tev, which solves the hierarchy problem. We will not consider this formalism in the following, and we refer the reader for example to refs [4, 5].

[3]At generic points in the field space $N = 2$ supersymmetry is completely broken, but at some particular points it can be either partially broken to $N = 1$ or completely unbroken [6].

type IIB, the R-R field strengths are a 1-form, a 3-form, and a 5-form, and the corresponding fluxes are found in the odd cohomologies. It can be checked by a KK reduction that the fluxes are correctly mapped [7].

The case of the NS-NS fluxes is less clear. Since the only NS-NS form is a 3-form field strength in both type IIA and IIB, the form fluxes should be found in the odd cohomologies in both cases, and there is no possible mirror map. However, the mirror fluxes may lie in the other fields of the NS-NS sector, the metric and the dilaton. This has led Vafa to conjecture that the mirror manifold should no longer be Calabi-Yau, but should instead have a non-integrable complex structure, and the fluxes would be expected to lead to this deformation of the geometry [8]. In this thesis we will introduce new manifolds, called half-flat spaces, which we conjecture to be the mirror image of a Calabi-Yau manifold when NS-NS fluxes are turned on. We will display several checks for this conjecture, which are based on the papers [9] and [10].

Compactifications of $N = 2$ $d = 10$ supergravities will be the subject of the first two chapters. These supergravities bear invariance under 32 supercharges. We will consider compactifications on general manifolds with $SU(3)$-structure (which includes Calabi-Yau), and consequently we will obtain (gauged or ungauged) $N = 2$ supergravities in 4 dimensions. Such theories contain the gravity multiplet, as well as vector multiplets and hypermultiplets in the matter sector. The couplings of these multiplets are characterized by two holomorphic homogeneous functions of degree 2, the prepotentials $\mathcal{F}(X)$ and $\mathcal{F}(z)$, one describing each sector. The dynamics of these theories is highly constrained by supersymmetry, but there is still room for the choice of a prepotential. Supergravities with $N > 2$ are of course more constrained, and the maximal N for which matter multiplets exist is $N = 4$. Thus $N = 4$ supergravities are of particular interest. Their dynamics allows for matter multiplets, but their structure is quite simple since entirely determined by supersymmetry. Although the action for this multiplet is known for quite long [11,12], until recently a completely satisfactory formulation in superspace was still missing. In [13], a set of constraints on the torsion was proposed, and it was argued that the resulting multiplet should be equivalent to the one of [12]. The particular features of this formulation were that it was making use of central charge superspace, and the vectors (graviphotons) were identified in the components of the vielbein carrying central charge indices. However, a complete proof of the equivalence between the two formalisms, in components and in superspace, was not yet worked out. In this thesis, we will fill in this gap. We will compute the equations of motion for all members of the multiplet in the superspace formalism, and we will show that they are exactly the same as the one arising from the Lagrangian of [12]. These results were obtained in [14].

In the second chapter, we describe in more detail the procedure of compactification, in general, and then in the particular case of a Calabi-Yau manifold. The main features of these compactifications in the case of type II theories are reviewed, the stress is put on the notion of moduli space, and as a conclusion we display the map between type IIA and type IIB supergravities, illustrating mirror symmetry.

In the third chapter, we introduce the half-flat manifolds. We recall their properties relevant to our purpose, and we motivate our conjecture by compactifying type IIA supergravity on such manifolds and showing that it is equivalent to type IIB supergravity on a Calabi-Yau manifold with NS-NS fluxes turned on. We then check the converse of the above procedure. We compactify type IIB theory on a half-flat manifold, and we show that it is equivalent to type IIA on a Calabi-Yau manifold with NS-NS fluxes turned on. We conclude with some conjectures about the moduli space of half-flat manifolds and we try to give hints about possible ansätze for a description of the magnetic fluxes.

In chapter 4, we present our work on $N = 4$ supergravity in 4 dimensions. We recall how the use of central charge superspace made it possible to identify the gravity multiplet in the components of the vielbein and the torsion, and we show the equivalence with the formulation in components by deriving the equations of motions for all members of the multiplet and identifying them with the ones obtained from the Lagrangian of [12].

Various elementary notions of differential geometry can be found in appendix A. Appendix B contains basic properties of Calabi-Yau manifolds, and many formulae useful for the compactification are gathered. In appendix C we briefly recall the compactification of type II theories on Calabi-Yau manifolds with NS fluxes. In appendix D we review a few facts about G-structures from the mathematical point of view. We present the computation of the Ricci scalar of half-flat manifolds in appendix E. Finally, appendix F displays the components of the torsion and curvature for $N = 4$ supergravity in central charge superspace.

Chapter 2

Calabi-Yau compactifications

This section is a review of the compactification of type II supergravities on Calabi-Yau manifolds. Such compactifications were first considered in [15] for type IIA and in [16] for type IIB, see [17] for a review in more recent notations.

In the first part we recall briefly the basics of the theory of reduction à la Kaluza-Klein, first on the circle, and then on a general n-dimensional manifold. We emphasize the main features of such reductions, and the relations between the topology of the internal manifold and the properties of the 4-dimensional theory. Then we deal with the issue of supersymmetry conservation during compactification which leads to the emergence of Calabi-Yau manifolds. We describe in details the reduction of the Ricci scalar and we introduce the notion of moduli space. In the second and the third parts, we turn to the compactification of the bosonic actions for type IIA and IIB supergravities. We show how the fields arrange in supergravity multiplets and we give the 4-dimensional action. We conclude by displaying the mirror map between the two theories.

2.1 Kaluza-Klein compactification

2.1.1 Reduction on a circle

In the case of the compactification on a circle, the d space-time coordinates $\hat{x}^M$, split into one set of $d-1$ space-time coordinate x^μ and one internal coordinate y, subject to the periodicity condition $y \sim y + R$ where R is the radius of the circle. It is well known that with such a periodicity property, any quantity $\hat{\Phi}$ can be expanded on a basis of periodic functions

$$\hat{\Phi}(\hat{x}) = \sum_n \tilde{\Phi}_n(x) \mathrm{e}^{in\frac{y}{R}} \tag{2.1}$$

where $\tilde{\Phi}_n(x)$ is the Fourrier transform of $\hat{\Phi}(\hat{x})$. We note that the $\mathrm{e}^{in\frac{y}{R}}$ are solutions to Laplace equation

$$\Delta \mathrm{e}^{in\frac{y}{R}} = \frac{\partial^2}{\partial y^2} \mathrm{e}^{in\frac{y}{R}} = -\frac{n^2}{R^2} \mathrm{e}^{in\frac{y}{R}} \tag{2.2}$$

with "mass" n/R. In all that follows, we will only consider the low energy limits of string theory, supergravities. Thus we will always truncate the summation at the massless level, which means that we only keep the term with $n = 0$ in (2.2), called massless mode. For more details about the consistency of this procedure, see [18] p.85 and references therein. As an example, we display the massless reduction of the metric

$$\hat{g}_{MN} \longrightarrow \begin{pmatrix} g_{\mu\nu} & V_\mu \\ V_\mu & \phi \end{pmatrix}. \tag{2.3}$$

When the two indices are external, the d-dimensional metric becomes the $(d-1)$-dimensional one, when one index is external and one is internal, it has the index structure of a space-time vector V_μ, and when the two indices are internal, it is a scalar ϕ. None of these new fields depends on the internal coordinate y, which corresponds to a massless expansion. This leads, up to field redefinitions, to the following expansion for the Ricci scalar

$$\hat{\mathcal{R}} = \mathcal{R} + F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi\partial^\mu\phi \tag{2.4}$$

where $F_{\mu\nu}$ is the field strength of the vector V_μ. This is the action for gravity coupled to electro-magnetism and an uncharged scalar.

2.1.2 Reduction on a compact manifold of dimension n

Kaluza-Klein reductions applied to supergravity have been described in details in [19, 20]. Here we only give the features that will be relevant to the next sections. The d space-time coordinates $\hat{x}^M$, split into one set of $d-n$ space-time coordinates x^μ and one set of n internal coordinates y^m. Let us take the example of a scalar field $\hat{\Phi}(\hat{x})$. Suppose that the metric is block-diagonal, which will always be the case from now on. Then the d-dimensional equation of motion can be written as

$$\hat{\Delta}\hat{\Phi} = m_d^2\hat{\Phi}(\hat{x}) = \Delta_{d-n}\hat{\Phi} + \Delta_n\hat{\Phi}. \tag{2.5}$$

where Δ_{d-n} and Δ_n are Laplacians in lower dimensions. The Kaluza-Klein ansatz on $\hat{\Phi}$ reads

$$\hat{\Phi}(\hat{x}^M) = \phi^i(x^\mu)\omega_i(y^m) \tag{2.6}$$

where ω_i is a set of a priori unknown functions, counted by the index i. We assume that the $(d-n)$-dimensional scalars ϕ^i also obey their usual field equation, which leads to

$$m_d^2\phi^i(x^\mu)\omega_i(y^m) = m_{d-n}^2\phi^i(x^\mu)\omega_i(y^m) + \phi^i(x^\mu)\Delta_n[\omega_i(y^m)]. \tag{2.7}$$

This means that the functions ω_i have to satisfy Laplace equation

$$\Delta_n\omega_i = (m_d^2 - m_{d-n}^2)\omega_i. \tag{2.8}$$

A result of differential analysis states that on a compact manifold, the eigenvalues of Laplace operator Δ are of the form n/S where S corresponds to the size of the manifold, and n is an integer. Again, we will only consider massless expansions, for which $n=0$. The unknown functions are thus harmonic

$$\Delta\omega_i = 0. \tag{2.9}$$

The bosonic fields of supergravities are either the metric or forms. The case of the metric is described in section 2.1.4. For the forms, the above result generalizes in the following way. A p-form $\hat{B}_p$ is expanded on all harmonic q-forms for $0 \leq q \leq p$

$$\hat{B}_p = B_p^{i_0}\omega_{i_0} + B_{p-1}^{i_1}\omega_{i_1} + ... + B_0^{i_p}\omega_{i_p} \tag{2.10}$$

where ω_{i_0} is a basis for the harmonic 0-forms and so on. For the definition of harmonicity on forms, see appendix A.2.

2.1.3 Calabi-Yau requirement

Consider now a supergravity theory in 10 dimensions, compactified on a 6-dimensional compact space. On the fermionic side, there is always a gravitino, whose supersymmetry transformation is related to the covariant derivative of the parameter. Suppose we are looking for a supersymmetric

background, then all vacuum values of transformations of fermionic fields must vanish. For the gravitino, we obtain

$$< \delta\hat{\Psi}_M >=< \mathcal{D}_M \hat{\epsilon} >= 0, \tag{2.11}$$

where $\hat{\epsilon}$ is the supersymmetry parameter. We use the Kaluza-Klein ansatz for this spinor

$$\hat{\epsilon}(\hat{x}) = \epsilon(x)\eta(y) \tag{2.12}$$

where ϵ is a spinor in 4 dimensions, and η is a spinor on the internal space. If we take an internal component in (2.11), we obtain that η must be covariantly constant

$$\nabla_m \eta = 0. \tag{2.13}$$

This is a very strong statement. For each covariantly constant spinor on the internal space, there is one conserved supersymmetry in 4 dimensions. For type II theories, the Kaluza-Klein ansatz reads

$$\hat{\epsilon}^A(\hat{x}) = \epsilon^A(x)\eta(y) \tag{2.14}$$

where $A = 1, 2$ counts the supersymmetry operators. Since we are interested in $N = 2$ supergravities in 4 dimensions, we must compactify on a space possessing one covariantly constant spinor: such spaces are called Calabi-Yau manifolds. These manifolds have numerous interesting properties. The fact that they admit one covariantly constant spinor restricts their holonomy group to $SU(3)$. They are complex, Kähler and Ricci-flat. This is consistent with Einstein's equation which relates the Ricci tensor to squares of field strengths of forms appearing in the spectra of type II supergravities

$$\hat{R}_{MN} \sim \hat{F}_{MPQ\ldots}\hat{F}_N{}^{PQ\ldots}. \tag{2.15}$$

In the case of KK compactifications, the field strengths have no purely internal components, which means that to be a solution to the equations of motions, the internal space must be Ricci-flat. Among other properties, Calabi-Yau manifolds have one and only one covariantly constant harmonic (3,0)-form, and their Hodge diamond has the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{l} b^0 = 1 \\ b^1 = 0 \\ b^2 = h^{(1,1)} \\ b^3 = 2 + 2h^{(2,1)} \\ b^4 = h^{(1,1)} \\ b^5 = 0 \\ b^6 = 1. \end{array} \tag{2.16}$$

2.1.4 Moduli space

We have seen in section 2.1.2 that the general ansatz for the compactification of forms requires expansion on harmonic forms on the Calabi-Yau manifold. This is in some sense also true for the metric. When the two indices are external, we obtain the 4-dimensional metric. There will be no components with one external and one internal index, because there are no 1-forms on the Calabi-Yau (2.16). Now we are left with purely internal components. Solutions to supergravities are generally not isolated, but come in continuous families, parameterized by the moduli. The metric moduli are thus infinitesimal deformations of the internal metric that conserve the Calabi-Yau conditions. Let us start from a background with hermitian metric g^0_{mn} and define the metric on a Calabi-Yau manifold infinitesimally close to the first one by

$g_{mn} = g^0_{mn} + \delta g_{mn}$. The constraint that the new manifold is Calabi-Yau can be expressed by imposing Ricci-flatness. This gives Lichnerowicz equation

$$\nabla^l \nabla_l \delta g_{mn} - [\nabla^l \, , \, \nabla_m] \delta g_{ln} - [\nabla^l \, , \, \nabla_n] \delta g_{lm} = 0. \tag{2.17}$$

Considering the properties of the Riemann tensor of a Kähler space A.5, we can see that, in complex indices, this equation splits into one on the mixed part of the metric, and one on the pure part. In the coordinates (A.38), the Kähler class is directly related to the metric

$$J_{\alpha\bar{\alpha}} = i g_{\alpha\bar{\alpha}}, \tag{2.18}$$

so the mixed variations of the metric are Kähler class deformations. Suppose we apply to the metric a pure variation followed by a small change of coordinates f^m in such a way that the first variation is annihilated. Then we have

$$\delta g_{\alpha\beta} = \frac{\partial \bar{f}^{\bar{\alpha}}}{\partial z^{\alpha}} g_{\bar{\alpha}\beta} + \frac{\partial \bar{f}^{\bar{\alpha}}}{\partial z^{\beta}} g_{\bar{\alpha}\alpha}. \tag{2.19}$$

This means that f cannot be holomorphic and the manifold obtained after a pure variation $\delta g_{\alpha\beta}$ has a different complex structure. This is why such deformations are complex structure deformations.

Obviously, (2.17) is Laplace equation (A.25), except that δg_{mn} is not a form. Following this idea, we expand the mixed component of δg on the (1,1)-forms on the Calabi-Yau

$$\delta g_{\alpha\bar{\beta}} = -i v^i (\omega_i)_{\alpha\bar{\beta}} \tag{2.20}$$

where v^i are $h^{(1,1)}$ real scalars. ω_i is harmonic, so it satisfies Laplace equation, and (2.20) solves the mixed part of Lichnerowicz equation. Since there are no (2,0)-forms, and anyway $\delta g_{\alpha\beta}$ is symmetric, it is not possible to expand it directly. We take instead

$$\delta g_{\alpha\beta} = \frac{i}{||\Omega||^2} \bar{z}^a (\bar{\eta}_a)_{\alpha\bar{\beta}\bar{\gamma}} \Omega^{\bar{\beta}\bar{\gamma}}{}_{\beta}, \tag{2.21}$$

where Ω is the (3,0)-form and we have expanded on the $(1,2)$-forms $\bar{\eta}_a$, with $h^{(2,1)}$ scalar complex coefficients $\bar{z}^a$. It is shown in appendix B.3 that this is symmetric and solution to (2.17). The metric moduli space is thus generated by $h^{(1,1)} + 2h^{(2,1)}$ real parameters. For further information about its structure, see appendix B.5.

We also need to compute the Ricci scalar in the 10-dimensional action. Since the moduli are infinitesimal parameters, we make a perturbation expansion, and we keep all terms up to order 2. The detailed calculation can be found in appendix B.4. Here we only display the result, which, as might be expected from (2.4), shows the emergence of kinetic terms for the scalars v^i , z^a

$$\int \mathrm{d}^{10}\hat{x} \sqrt{-\hat{g}} \hat{R} = \int \mathrm{d}^4 x \sqrt{-g_4} \left(\mathcal{K} R_4 + P_{ij} \partial_\mu v^i \partial^\mu v^j + Q_{ab} \partial_\mu z^a \partial^\mu \bar{z}^b \right) \tag{2.22}$$

where $\mathcal{K}$ is the volume of the Calabi-Yau manifold and the couplings are defined in (B.69) and (B.70). The scalars are organized in two non-linear sigma models, whose metrics are both Special Kähler with Kähler potentials (B.82) and (B.93). The whole moduli space has the structure of a product of two Special Kähler manifolds, one corresponding to the Kähler class deformations $\mathcal{M}_{1,1}$, of complex[1] dimension $h^{(1,1)}$, and one corresponding to the complex structure deformations $\mathcal{M}_{2,1}$ of complex dimension $h^{(2,1)}$

$$\mathcal{M} = \mathcal{M}_{1,1} \times \mathcal{M}_{2,1}. \tag{2.23}$$

[1] Once the B_2 moduli are taken into account as in (B.79).

2.2 Compactification of type IIA supergravity

2.2.1 Reduction of the Ricci scalar and the dilaton

As we will see in the next sections, there is a part of the Lagrangian which is common to type IIA and type IIB supergravities. This comes from the fact that the spectrum of both theories is composed of two sets of fields, the R-R and the NS-NS fields. Type IIA and type IIB supergravities have the same NS-NS spectrum, but differ in the R-R sector. Thus their NS-NS Lagrangian is identical. It contains the graviton $\hat{g}_{MN}$, the dilaton $\hat{\phi}$, and the NS 2-form $\hat{B}_2$. Later on in this thesis, we will be interested in turning on fluxes for some forms, including $\hat{B}_2$, such that the part of the Lagrangian which contains only the graviton and the dilaton will always be compactified in the same way. This procedure is described below. The action we will study is

$$S = \int e^{-2\hat{\phi}} \left(-\frac{1}{2}\hat{R} *\mathbf{1} + 2d\hat{\phi} \wedge *d\hat{\phi} \right). \tag{2.24}$$

Remark that the kinetic term for the scalars has a wrong sign. This is because this action is written in the string frame. We will now perform a Weyl rescaling on (2.24) to go to Einstein's frame. Recall also that formula (B.68) is only true up to total derivatives, which means that it is not possible to use it directly as it stands in the string frame.

1$^{\text{st}}$ step : going to Einstein's frame

We perform the Weyl rescaling (A.8) with $\Omega = \mathrm{e}^{-\hat{\phi}/4}$. We obtain

$$S = \int -\frac{1}{2}\hat{R} *\mathbf{1} - \frac{1}{4}d\hat{\phi} \wedge *d\hat{\phi}. \tag{2.25}$$

Here we have to be careful about the fact that under this rescaling, the determinant of the metric and the inverse metrics used to contract indices are not written explicitly, but should be transformed. Remark that now the kinetic term for the dilaton has a correct sign.

2$^{\text{nd}}$ step : compactifying

We use formula (B.68). This leads to

$$\begin{aligned} S = \int & -\frac{1}{2}\mathcal{K}R *\mathbf{1} - \mathcal{K}\frac{1}{4}d\hat{\phi} \wedge *d\hat{\phi} - \frac{1}{2}P_{ij}dv^i \wedge *dv^j \\ & -\frac{1}{2}Q_{ab}dz^a \wedge *d\bar{z}^b, \end{aligned} \tag{2.26}$$

where the integral is now only on the 4-dimensional space-time.

3$^{\text{rd}}$ step : Weyl rescaling for the volume $\mathcal{K}$

In order to have the usual normalization for the Ricci scalar, we perform the Weyl rescaling (A.8) with $\Omega = \mathcal{K}^{\frac{1}{2}}$. We obtain

$$\begin{aligned} S = \int & -\frac{1}{2}R *\mathbf{1} - \frac{3}{4}d\ln\mathcal{K} \wedge *d\ln\mathcal{K} - \frac{1}{4}d\hat{\phi} \wedge *d\hat{\phi} - \frac{1}{2\mathcal{K}}P_{ij}dv^i \wedge *dv^j \\ & -\frac{1}{2\mathcal{K}}Q_{ab}dz^a \wedge *d\bar{z}^b. \end{aligned} \tag{2.27}$$

4^{th} step : rotation of the v^i

We realize now a rotation of the v^i. The purpose is to eliminate the term $d\ln\mathcal{K}\wedge *d\ln\mathcal{K}$ which is not a new scalar, but depends on the v^i. We define

$$v^i = \mathrm{e}^{-\frac{1}{2}\hat{\phi}}\tilde{v}^i. \tag{2.28}$$

Considering that the basis forms are independent of v^i, we find the following transformation rules for the integrals defined in appendix B.2

$$\begin{aligned}
\mathcal{K}_{ijk} = \tilde{\mathcal{K}}_{ijk} \quad &; \quad \mathcal{K}_{ij} = \mathrm{e}^{-\frac{1}{2}\hat{\phi}}\tilde{\mathcal{K}}_{ij} \\
P_{ij} = \mathrm{e}^{-\frac{1}{2}\hat{\phi}}\tilde{P}_{ij} \quad &; \quad \mathcal{K}_i = \mathrm{e}^{-\hat{\phi}}\tilde{\mathcal{K}}_i \\
\mathcal{K} = \mathrm{e}^{-\frac{3}{2}\hat{\phi}}\tilde{\mathcal{K}} \quad &; \quad g_{\alpha\bar{\beta}} = \mathrm{e}^{-\frac{1}{2}\hat{\phi}}\tilde{g}_{\alpha\bar{\beta}},
\end{aligned} \tag{2.29}$$

the last equation holding because the volume is an integral on $\sqrt{g_6}$ which is of order 3 in the metric with lower indices. Performing a careful counting of the number of lower metrics in Q_{ab}, we find

$$Q_{ab} = \mathrm{e}^{-\frac{3}{2}\hat{\phi}}\tilde{Q}_{ab}. \tag{2.30}$$

The transformation of the kinetic terms for the v^i is

$$\frac{1}{2\mathcal{K}}P_{ij}\partial v^i\partial v^j \;=\; \tilde{g}_{ij}\partial\tilde{v}^i\partial\tilde{v}^j - \frac{1}{2}\partial\ln\tilde{\mathcal{K}}\partial\ln\tilde{\mathcal{K}} + \frac{5}{4}\partial\ln\tilde{\mathcal{K}}\partial\hat{\phi} - \frac{15}{16}\partial\hat{\phi}\partial\hat{\phi} \tag{2.31}$$

where the metric g_{ij} is given in (B.22) and we have used

$$\tilde{\mathcal{K}}_i\partial\tilde{v}^i = 2\partial\tilde{\mathcal{K}}. \tag{2.32}$$

Finally we obtain the action

$$\begin{aligned}
S \;=\; \int & -\frac{1}{2}R *\mathbf{1} - d\phi\wedge *d\phi - g_{ij}dv^i\wedge *dv^j \\
& - g_{ab}dz^a\wedge *d\bar{z}^b.
\end{aligned} \tag{2.33}$$

Here we defined the 4-dimensional dilaton by

$$\phi = \hat{\phi} - \frac{1}{2}\ln\tilde{\mathcal{K}}, \tag{2.34}$$

we dropped the tildes, and the metric for the scalars z^a is

$$g_{ab} = \frac{1}{2\mathcal{K}}Q_{ab}. \tag{2.35}$$

The metrics g_{ij} and g_{ab} exhibit properties detailed in appendix B.5.

2.2.2 Matter part of type IIA supergravity

In this section we recall the known results of type IIA supergravity compactified on a Calabi-Yau threefold Y. The NS-NS spectrum consists of the graviton, the dilaton, a 2-form $\hat{B}_2$, and the R-R spectrum contains a 1-form $\hat{A}_1$ and a 3-form $\hat{C}_3$. We start from the following action in 10 dimensions

$$\begin{aligned} S &= \int e^{-2\hat{\phi}}\left(-\frac{1}{2}\hat{R}*\mathbf{1}+2d\hat{\phi}\wedge *d\hat{\phi}-\frac{1}{4}\hat{H}_3\wedge *\hat{H}_3\right) \\ &\quad -\frac{1}{2}\int\left(\hat{F}_2\wedge *\hat{F}_2+\hat{F}_4\wedge *\hat{F}_4\right)+\frac{1}{2}\int \hat{H}_3\wedge\hat{C}_3\wedge d\hat{C}_3\,, \end{aligned} \tag{2.36}$$

where

$$\hat{H}_3 = d\hat{B}_2\ , \qquad \hat{F}_2 = d\hat{A}_1\ , \qquad \hat{F}_4 = d\hat{C}_3 - \hat{A}_1\wedge\hat{H}_3, \tag{2.37}$$

and we will follow the above procedure step by step. From now on we will not display the part with the graviton and the dilaton whose behaviour has been studied in the previous section.

$1^{\rm st}$ step : going to the Einstein's frame

We perform the Weyl rescaling (A.8) with $\Omega = \mathrm{e}^{-\hat{\phi}/4}$. We obtain

$$\begin{aligned} S &= -\frac{1}{4}\int \mathrm{e}^{-\hat{\phi}}\hat{H}_3\wedge *\hat{H}_3-\frac{1}{2}\int \mathrm{e}^{\frac{3}{2}\hat{\phi}}\hat{F}_2\wedge *\hat{F}_2-\frac{1}{2}\int \mathrm{e}^{\frac{1}{2}\hat{\phi}}\hat{F}_4\wedge *\hat{F}_4 \\ &\quad +\frac{1}{2}\int \hat{H}_3\wedge\hat{C}_3\wedge d\hat{C}_3\,, \end{aligned} \tag{2.38}$$

where the Chern-Simons term remains unchanged because it does not contain any metric.

$2^{\rm nd}$ step : expanding

In the KK reduction we expand the ten-dimensional fields in terms of harmonic forms on Y

$$\begin{aligned} \hat{A}_1 &= A^0\ , \\ \hat{C}_3 &= C_3 + A^i\wedge\omega_i + \xi^A\alpha_A + \tilde{\xi}_A\beta^A\ , \\ \hat{B}_2 &= B_2 + b^i\omega_i\ , \end{aligned} \tag{2.39}$$

where C_3 is a 3-form, B_2 a 2-form, (A^0, A^i) are 1-forms and $b^i, \xi^A, \tilde{\xi}_A$ are scalar fields in $D = 4$. The ω_i are a basis for the harmonic $(1,1)$-forms and $(\alpha_A\ , \beta^A)$ a basis for the harmonic 3-forms, see appendix B.2. All these 4-dimensional fields are organized in supergravity multiplets. The gravity multiplet contains the metric $g_{\mu\nu}$ and the graviphoton A^0. The $h^{(1,1)}$ vectors A^i, together with the $2h^{(1,1)}$ scalars v^i, b^i belong to $h^{(1,1)}$ vector multiplets. The rest of the fields only consists of scalars. Therefore all these scalars belong to hypermultiplets. Indeed, C_3 is non dynamical in 4 dimensions, and B_2 is dual to a scalar a. Collecting the remaining $4h^{(2,1)} + 4$ scalars ϕ, z^a, a, ξ^A and $\tilde{\xi}_A$, we obtain $h^{(2,1)} + 1$ hypermultiplets. Since the harmonic forms are closed, the differential operator d acts only on the space-time forms

$$d\hat{A}_1 = dA^0 \tag{2.40}$$

$$d\hat{C}_3 = dC_3 + dA^i\wedge\omega_i + d\xi^A\wedge\alpha_A + d\tilde{\xi}_A\wedge\beta^A \tag{2.41}$$

$$\hat{H}_3 = H_3 + db^i\wedge\omega_i. \tag{2.42}$$

The terms of (2.38) are thus expanded according to

$$
\begin{aligned}
-\frac{1}{4}\mathrm{e}^{-\hat{\phi}}\int_Y \hat{H}_3\wedge *\hat{H}_3 &= -\frac{\mathcal{K}}{4}\mathrm{e}^{-\hat{\phi}}\, H_3\wedge *H_3 - \mathcal{K}\mathrm{e}^{-\hat{\phi}} g_{ij} db^i\wedge *db^j &(2.43)\\
-\frac{1}{2}\mathrm{e}^{\frac{3}{2}\hat{\phi}}\int_Y \hat{F}_2\wedge *\hat{F}_2 &= -\frac{\mathcal{K}}{2}\mathrm{e}^{\frac{3}{2}\hat{\phi}}\, dA^0\wedge *dA^0\ , &(2.44)\\
-\frac{1}{2}\mathrm{e}^{\frac{1}{2}\hat{\phi}}\int_Y \hat{F}_4\wedge *\hat{F}_4 &= -\frac{\mathcal{K}}{2}\mathrm{e}^{\frac{1}{2}\hat{\phi}}\,(dC_3 - A^0\wedge H_3)\wedge *(dC_3 - A^0\wedge H_3)\\
&\quad -2\mathcal{K}\mathrm{e}^{\frac{1}{2}\hat{\phi}} g_{ij}(dA^i - A^0 db^i)\wedge *(dA^j - A^0 db^j)\\
&\quad +\frac{1}{2}\mathrm{e}^{\frac{1}{2}\hat{\phi}}\left(\mathrm{Im}\,\mathcal{M}^{-1}\right)^{AB}\left[d\tilde{\xi}_A + \mathcal{M}_{AC}d\xi^C\right]\wedge *\left[d\tilde{\xi}_B + \bar{\mathcal{M}}_{BD}d\xi^D\right]\ ,\\
\frac{1}{2}\int_Y \hat{H}_3\wedge \hat{C}_3\wedge d\hat{C}_3 &= -\frac{1}{2}H_3\wedge(\xi^A d\tilde{\xi}_A - \tilde{\xi}_A d\xi^A) + \frac{1}{2}db^i\wedge A^j\wedge dA^k\mathcal{K}_{ijk}\ . &(2.45)
\end{aligned}
$$

where the integration on the internal manifold has been carried out and the matrix $\mathcal{M}$ is defined in appendix B.5.

3rd step : Weyl rescaling for the volume $\mathcal{K}$

In order to recover a standard kinetic term for gravity, we need to perform a Weyl rescaling with Weyl factor $\mathcal{K}^{\frac{1}{2}}$

$$
\begin{aligned}
-\frac{1}{4}\mathrm{e}^{-\hat{\phi}}\int_Y \hat{H}_3\wedge *\hat{H}_3 &= -\frac{\mathcal{K}^2}{4}\mathrm{e}^{-\hat{\phi}}\, H_3\wedge *H_3 - \mathrm{e}^{-\hat{\phi}} g_{ij} db^i\wedge *db^j &(2.46)\\
-\frac{1}{2}\mathrm{e}^{\frac{3}{2}\hat{\phi}}\int_Y \hat{F}_2\wedge *\hat{F}_2 &= -\frac{\mathcal{K}}{2}\mathrm{e}^{\frac{3}{2}\hat{\phi}}\, dA^0\wedge *dA^0\ , &(2.47)\\
-\frac{1}{2}\mathrm{e}^{\frac{1}{2}\hat{\phi}}\int_Y \hat{F}_4\wedge *\hat{F}_4 &= -\frac{\mathcal{K}^3}{2}\mathrm{e}^{\frac{1}{2}\hat{\phi}}\,(dC_3 - A^0\wedge H_3)\wedge *(dC_3 - A^0\wedge H_3)\\
&\quad -2\mathcal{K}\mathrm{e}^{\frac{1}{2}\hat{\phi}} g_{ij}(dA^i - A^0 db^i)\wedge *(dA^j - A^0 db^j)\\
&\quad +\frac{1}{2\mathcal{K}}\mathrm{e}^{\frac{1}{2}\hat{\phi}}\left(\mathrm{Im}\,\mathcal{M}^{-1}\right)^{AB}\left[d\tilde{\xi}_A + \mathcal{M}_{AC}d\xi^C\right]\wedge *\left[d\tilde{\xi}_B + \bar{\mathcal{M}}_{BD}d\xi^D\right]\ ,\\
\frac{1}{2}\int_Y \hat{H}_3\wedge \hat{C}_3\wedge d\hat{C}_3 &= -\frac{1}{2}H_3\wedge(\xi^A d\tilde{\xi}_A - \tilde{\xi}_A d\xi^A) + \frac{1}{2}db^i\wedge A^j\wedge dA^k\mathcal{K}_{ijk}\ . &(2.48)
\end{aligned}
$$

4th step : rotation of the v^i

Counting the powers of the metric in the definition of $\mathcal{M}$ as an integral, we can deduce that $\mathcal{M}$ is invariant under this rotation. Once everything is written in terms of the 4-dimensional dilaton, we obtain

$$
\begin{aligned}
-\frac{1}{4}\mathrm{e}^{-\hat{\phi}}\int_Y \hat{H}_3\wedge *\hat{H}_3 &= -\frac{1}{4}\mathrm{e}^{-4\phi}\,H_3\wedge *H_3 - g_{ij}db^i\wedge *db^j && (2.49)\\
-\frac{1}{2}\mathrm{e}^{\frac{3}{2}\hat{\phi}}\int_Y \hat{F}_2\wedge *\hat{F}_2 &= -\frac{\mathcal{K}}{2}\,dA^0\wedge *dA^0\ , && (2.50)\\
-\frac{1}{2}\mathrm{e}^{\frac{1}{2}\hat{\phi}}\int_Y \hat{F}_4\wedge *\hat{F}_4 &= -\frac{\mathcal{K}}{2}\mathrm{e}^{-4\phi}\,(dC_3 - A^0\wedge H_3)\wedge *(dC_3 - A^0\wedge H_3)\\
&\quad -2\mathcal{K}g_{ij}(dA^i - A^0db^i)\wedge *(dA^j - A^0db^j)\\
&\quad +\frac{1}{2}\mathrm{e}^{2\phi}\left(\mathrm{Im}\,\mathcal{M}^{-1}\right)^{AB}\left[d\tilde{\xi}_A + \mathcal{M}_{AC}d\xi^C\right]\wedge *\left[d\tilde{\xi}_B + \bar{\mathcal{M}}_{BD}d\xi^D\right]\ ,\\
\frac{1}{2}\int_Y \hat{H}_3\wedge\hat{C}_3\wedge d\hat{C}_3 &= -\frac{1}{2}H_3\wedge(\xi^A d\tilde{\xi}_A - \tilde{\xi}_A d\xi^A) + \frac{1}{2}db^i\wedge A^j\wedge dA^k\mathcal{K}_{ijk}\,. && (2.51)
\end{aligned}
$$

With these expressions we can now combine the different terms appearing in the action (2.36) The dualization of the 3-form C_3 in 4 dimensions produces a contribution to the cosmological constant. As shown in [21] this constant can be viewed as a specific RR-flux. Since we are not interested in RR-fluxes here we choose it to be zero and hence discard the contribution of C_3 in 4 dimensions. Thus the only thing we still need to do in order to recover the standard spectrum of $N = 2$ supergravity in 4 dimensions is to dualize the 2-form B_2 to the axion a. The action for B_2 is

$$\mathcal{L}_{H_3} = -\frac{1}{4}e^{-4\phi}H_3\wedge *H_3 + \frac{1}{2}H_3\wedge\left(\tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A\right). \tag{2.52}$$

Counting the degrees of freedom of B_2, we know that it should be dual to a scalar a, called the axion. We add to (2.52) the term

$$+\frac{1}{2}H_3\wedge da \tag{2.53}$$

which realizes the Bianchi identity of B_2 as an equation of motion for a. H_3 can consequently be considered as a fundamental field, and eliminated through its equation of motion

$$-\frac{1}{2}e^{-4\phi} * H_3 + \frac{1}{2}\left(da + \tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A\right) = 0. \tag{2.54}$$

The Lagrangian for a becomes

$$\mathcal{L}_a = -\frac{1}{4}e^{+4\phi}\left(da + \tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A\right)\wedge *\left(da + \tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A\right). \tag{2.55}$$

The usual $N = 2$ supergravity couplings can be read off after redefining the gauge fields $A^i \to A^i - b^iA^0$ and introducing the collective notation $A^I = (A^0, A^i)$ where $I = (0, i) = 0, \ldots, h^{(1,1)}$.

Collecting all terms from (2.49)-(2.51) and taking into account the gravity sector 2.33, we obtain

$$
\begin{aligned}
S_{IIA} \;=\; \int\Big[&-\frac{1}{2}R^*\mathbf{1} - g_{ij}dt^i\wedge *d\bar{t}^{\bar{j}} - h_{uv}dq^u\wedge *dq^v\\
&+\frac{1}{2}\,\mathrm{Im}\mathcal{N}_{IJ}F^I\wedge *F^J + \frac{1}{2}\,\mathrm{Re}\mathcal{N}_{IJ}F^I\wedge F^J\Big],
\end{aligned} \tag{2.56}
$$

where the gauge coupling matrix $\mathcal{N}$ is defined in appendix B.5, the scalars b^i coming from the NS 2-form and v^i from the Kähler class deformations are complexified into $t^i = b^i + iv^i$, and h_{uv} is the σ-model metric for the scalars in the hypermultiplets [22]

$$\begin{aligned} h_{uv}dq^u \wedge *dq^v &= d\phi \wedge *d\phi + g_{ab}dz^a \wedge *d\bar{z}^b \\ &+\frac{e^{4\phi}}{4}\left[da + (\tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A)\right] \wedge {}^*\left[da + (\tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A)\right] \\ &-\frac{e^{2\phi}}{2}\left(\mathrm{Im}\,\mathcal{M}^{-1}\right)^{AB}\left[d\tilde{\xi}_A + \mathcal{M}_{AC}d\xi^C\right] \wedge {}^*\left[d\tilde{\xi}_B + \bar{\mathcal{M}}_{BD}d\xi^D\right]. \end{aligned} \tag{2.57}$$

2.3 Compactification of type IIB supergravity

In this section we recall the KK compactification of type IIB supergravity on a Calabi-Yau 3-fold $\tilde{Y}$. The 10 dimensional bosonic spectrum of type IIB supergravity consists of the metric $\hat{g}$, the antisymmetric tensor field $\hat{B}_2$ and the dilaton $\hat{\phi}$ in the NS-NS sector, an axion $\hat{l}$, a 2-form $\hat{C}_2$ and a 4-form $\hat{A}_4$ with self-dual field strength $*\hat{F}_5 = \hat{F}_5$ in the R-R sector. No local covariant action can be written for this theory in 10 dimensions due to the self-duality of $\hat{F}_5$. Instead we use the action [3]

$$\begin{aligned} S^{(10)}_{IIB} &= \int e^{-2\hat{\phi}}\left(-\frac{1}{2}\hat{R}*\mathbf{1} + 2d\hat{\phi}\wedge *d\hat{\phi} - \frac{1}{4}d\hat{B}_2 \wedge *d\hat{B}_2\right) \\ &-\frac{1}{2}\int\left(d\hat{l}\wedge *d\hat{l} + \hat{F}_3 \wedge *\hat{F}_3 + \frac{1}{2}\hat{F}_5 \wedge *\hat{F}_5\right) \\ &-\frac{1}{2}\int \hat{A}_4 \wedge d\hat{B}_2 \wedge d\hat{C}_2, \end{aligned} \tag{2.58}$$

where

$$\hat{F}_3 = d\hat{C}_2 - \hat{l}\hat{H}_3 \tag{2.59}$$

$$\hat{F}_5 = d\hat{A}_4 - d\hat{B}_2 \wedge \hat{C}_2\ , \tag{2.60}$$

and impose the self-duality of $\hat{F}_5$ separately, at the level of the equations of motion. The compactification proceeds as usual, by following the steps of the above sections. However, we will skip the steps that are not essential to our purpose.

We expand the 10-dimensional quantities in terms of harmonic forms on the Calabi-Yau manifold as

$$\begin{aligned} \hat{B}_2 &= B_2 + b^i \wedge \omega_i\ , \qquad i = 1,\ldots,h^{(1,1)}\ , \\ \hat{C}_2 &= C_2 + c^i \wedge \omega_i\ , \\ \hat{A}_4 &= D_2^i \wedge \omega_i + \rho_i \wedge \tilde{\omega}^i + V^A \wedge \alpha_A - U_A \wedge \beta^A\ , \qquad A = 1,\ldots,h^{(2,1)}\ , \end{aligned} \tag{2.61}$$

where B_2, C_2, D_2^i are two-forms, V^A, U_A are one-forms and b^i, c^i, ρ_i are scalar fields in $D = 4$. Only half of the fields in the expansion of $\hat{A}_4$ are independent due to the self-duality of $\hat{F}_5$. We choose to keep ρ_i and V^A as independent fields. The 4-dimensional spectrum arranges as follows. The gravity multiplet contains the metric $g_{\mu\nu}$ and the graviphoton V^0. The $h^{(2,1)}$ vectors V^a, together with the $2h^{(2,1)}$ scalars z^a belong to $h^{(2,1)}$ vector multiplets. Again, the rest of the fields only consists of scalars which go to hypermultiplets. Indeed, C_2 and B_2 are dual to two scalars h_1, h_2. Collecting the remaining $4h^{(1,1)} + 4$ scalars $\phi, b^i, v^i, c^i, h_1, h_2, l, \rho^i$, we obtain $h^{(1,1)} + 1$ hypermultiplets. For the field strengths, this gives

$$\hat{H}_3 = H_3 + db^i \wedge \omega_i \tag{2.62}$$

$$d\hat{C}_2 = dC_2 + dc^i \wedge \omega_i \tag{2.63}$$

$$d\hat{A}_4 = dD_2^i \wedge \omega_i + d\rho_i \wedge \tilde{\omega}^i + F^A \wedge \alpha_A - G_A \wedge \beta^A.$$

where $F^A = dV^A$ and $G_A = dU_A$. For $\hat{F}_5$, this leads to

$$\begin{aligned} \hat{F}_5 &= F^A \wedge \alpha_A - G_A \beta^A + \left(dD_2^i - db^i \wedge C_2 - c^i H_3\right) \wedge \omega_i \\ &\quad + d\rho_i \wedge \tilde{\omega}^i - c^i db^j \wedge \omega_i \wedge \omega_j . \end{aligned} \tag{2.64}$$

The straightforward expansion reads

$$-\frac{1}{4}\int_Y \hat{H}_3 \wedge *\hat{H}_3 = -\frac{\mathcal{K}}{4} H_3 \wedge *H_3 - \mathcal{K} g_{ij} db^i \wedge *db^j \tag{2.65}$$

$$-\frac{1}{2}\int_Y d\hat{l} \wedge *d\hat{l} = -\frac{\mathcal{K}}{2} dl \wedge *dl \ , \tag{2.66}$$

$$\begin{aligned} -\frac{1}{2}\int_Y \hat{F}_3 \wedge *\hat{F}_3 &= -\frac{\mathcal{K}}{2} (dC_2 - lH_3) \wedge *(dC_2 - lH_3) \\ &\quad -2\mathcal{K} g_{ij} (dc^i - l db^i) \wedge *(dc^j - l db^j) \end{aligned} \tag{2.67}$$

$$\begin{aligned} -\frac{1}{4}\int_Y \hat{F}_5 \wedge *\hat{F}_5 &= +\frac{1}{4} Im\mathcal{M}^{-1} \left(\tilde{G} - \mathcal{M}\tilde{F}\right) \wedge * \left(\tilde{G} - \bar{\mathcal{M}}\tilde{F}\right) \\ &\quad - \mathcal{K} g_{ij} d\tilde{D}_2^i \wedge * d\tilde{D}_2^j - \frac{1}{16\mathcal{K}} g^{ij} d\tilde{\rho}_i \wedge * d\tilde{\rho}_j \end{aligned} \tag{2.68}$$

$$-\frac{1}{2}\int_Y \hat{A}_4 \wedge \hat{H}_3 \wedge d\hat{C}_2 = -\frac{1}{2}\mathcal{K}_{ijk} D_2^i \wedge db^j \wedge dc^k - \frac{1}{2}\rho_i \left(dB_2 \wedge dc^i + db^i \wedge dC_2\right) \tag{2.69}$$

with

$$d\tilde{D}_2^i = dD_2^i - db^i \wedge C_2 - c^i dB_2 \tag{2.70}$$

$$d\tilde{\rho}_i = d\rho_i - \mathcal{K}_{ikl} c^k db^l . \tag{2.71}$$

The self-duality of $\hat{F}_5$ implies that only half of the fields appearing in the expansion of $\hat{A}_4$ in (2.61) are independent. Thus the expansion above cannot be used directly. On the other hand, we cannot impose the self-duality

$$\hat{F}_5 = *\hat{F}_5 \tag{2.72}$$

in the action because the kinetic term $\hat{F}_5 \wedge *\hat{F}_5$ would vanish. In this thesis we want to show two different but equivalent strategies to discard half of the fields in $\hat{A}_4$. The first one starts by writing a general Lagrangian involving the remaining terms, and then identifying the reduced 10-dimensional equations of motion with the 4-dimensional ones calculated from the general Lagrangian. To make this more precise, we decide to discard first D_2^i, using the self-duality

$$d\tilde{D}_2^i = \frac{1}{4\mathcal{K}} g^{ij} * d\tilde{\rho}_j . \tag{2.73}$$

From looking at all possible terms, we can infer the following Lagrangian

$$\begin{aligned} \mathcal{L}_{inf} &= k_1 g^{ij} (d\rho_i - \mathcal{K}_{ikl} c^k db^l) \wedge *(d\rho_j - \mathcal{K}_{jpq} c^p db^q) \\ &\quad + k_3 d\rho_i \wedge (c^i dB_2 + db^i \wedge C_2) + k_4 \mathcal{K}_{ijk} c^i c^j dB_2 \wedge db^k . \end{aligned} \tag{2.74}$$

From the action (2.58), we derive the 10-dimensional equation of motion for $\hat{C}_2$, in the limit where $\hat{l} = 0$, $\hat{\phi}$ is constant and the metric is flat,

$$d * d\hat{C}_2 = \hat{F}_5 \wedge d\hat{B}_2 \tag{2.75}$$

where we have used (2.72). This equation has two components $(4,4)$ and $(2,6)$, where (e,i) means order e in space-time indices and order i in internal indices. The component $(4,4)$ reads (after integration over the internal manifold)

$$d * dc^i = -\frac{1}{16\mathcal{K}^2} g^{il} g^{pk} \mathcal{K}_{lpj} db^j \wedge *d\tilde{\rho}_k + \frac{1}{4\mathcal{K}} g^{ij} d\tilde{\rho}_j \wedge dB_2. \tag{2.76}$$

The kinetic term for c^i can be read off in (2.67), and the identification with the equation calculated from $\mathcal{L}_{inf}$ gives the values

$$k_1 = -\frac{1}{8\mathcal{K}} \quad ; \quad k_3 = -1 \quad ; \quad k_4 = -\frac{1}{2}. \tag{2.77}$$

After integration over the internal manifold, the component $(2,6)$ reads

$$\mathcal{K} d * dC_2 = d\tilde{\rho}_i \wedge db^i. \tag{2.78}$$

The identification with the equation calculated from $\mathcal{L}_{inf}$ gives again $k_3 = -1$. To deal with the vectors, we choose to display an other strategy. We take the Lagrangian from the expansions (2.68) and (2.69)

$$\mathcal{L}_{FA} = +\frac{1}{4} Im\mathcal{M}^{-1} (G - \mathcal{M}F) \wedge * \left(G - \bar{\mathcal{M}}F\right). \tag{2.79}$$

For the same reason as above, it is not possible to impose the self-duality of $\hat{F}_5$

$$*G = Re\mathcal{M} * F - Im\mathcal{M}F \tag{2.80}$$

$$G = Re\mathcal{M}F + Im\mathcal{M} * F \tag{2.81}$$

directly. First we add the total derivative

$$+\frac{1}{2} F^A \wedge G_A\,, \tag{2.82}$$

and we remark that the equation of motion of G_A is exactly (2.80), so the self-duality will be taken into account in a non-trivial way once G_A is eliminated with its equation of motion. We find

$$\mathcal{L}_{FA} = +\frac{1}{2} Im\mathcal{M}_{AB} F^A \wedge *F^B + \frac{1}{2} Re\mathcal{M}_{AB} F^A \wedge F^B. \tag{2.83}$$

Finally, after the Weyl rescaling of the volume and the rotation of v^i, the whole action is

$$
\begin{aligned}
S_{IIB}^{(4)} \;=\; & \int -\frac{1}{2}R*\mathbf{1} - g_{ab}dz^a\wedge *d\bar{z}^b - g_{ij}dt^i\wedge *d\bar{t}^j - d\phi\wedge *d\phi \\
& -\frac{1}{4}e^{-4\phi}dB_2\wedge *dB_2 - \frac{1}{2}e^{-2\phi}\mathcal{K}\left(dC_2 - ldB_2\right)\wedge *\left(dC_2 - ldB_2\right) \\
& -\frac{1}{2}\mathcal{K}e^{2\phi}dl\wedge *dl - 2\mathcal{K}e^{2\phi}g_{ij}\left(dc^i - ldb^i\right)\wedge *\left(dc^j - ldb^j\right) \\
& -\frac{e^{2\phi}}{8\mathcal{K}}g^{-1\,ij}\left(d\rho_i - \mathcal{K}_{ikl}c^k db^l\right)\wedge *\left(d\rho_j - \mathcal{K}_{jmn}c^m db^n\right) \\
& +\left(db^i\wedge C_2 + c^i dB_2\right)\wedge\left(d\rho_i - \mathcal{K}_{ijk}c^j db^k\right) + \frac{1}{2}\mathcal{K}_{ijk}c^i c^j dB_2\wedge db^k \\
& +\frac{1}{2}Re\mathcal{M}_{AB}F^A\wedge F^B + \frac{1}{2}Im\mathcal{M}_{AB}F^A\wedge *F^B\ .
\end{aligned}
\tag{2.84}
$$

Now we want to dualize the 2-forms C_2 and B_2 with scalar duals h_1 and h_2. We add first

$$
+dC_2\wedge dh_1 \tag{2.85}
$$

and the Lagrangian of interest for C_2 is

$$
\begin{aligned}
\mathcal{L}_{C_2} \;=\; & -\frac{1}{2}e^{-2\phi}\mathcal{K}\left(dC_2 - ldB_2\right)\wedge *\left(dC_2 - ldB_2\right) \\
& -b^i dC_2\wedge d\rho_i + dC_2\wedge dh_1.
\end{aligned}
\tag{2.86}
$$

We eliminate dC_2 with its equation of motion and we find

$$
\begin{aligned}
\mathcal{L}_{h_1} \;=\; & -\frac{1}{2\mathcal{K}}e^{2\phi}\left(dh_1 - b^i d\rho_i\right)\wedge *\left(dh_1 - b^j d\rho_j\right) \\
& +ldB_2\wedge\left(dh_1 - b^i d\rho_i\right).
\end{aligned}
\tag{2.87}
$$

Repeating the same procedure with B_2, we obtain the action for type IIB supergravity on a Calabi-Yau manifold

$$
\begin{aligned}
S_{IIB}^{(4)} \;=\; & \int -\frac{1}{2}R*\mathbf{1} - g_{ab}dz^a\wedge *d\bar{z}^b - g_{ij}dt^i\wedge *d\bar{t}^j - d\phi\wedge *d\phi \\
& -\frac{e^{2\phi}}{8\mathcal{K}}g^{-1\,ij}\left(d\rho_i - \mathcal{K}_{ikl}c^k db^l\right)\wedge *\left(d\rho_j - \mathcal{K}_{jmn}c^m db^n\right) \\
& -2\mathcal{K}e^{2\phi}g_{ij}\left(dc^i - ldb^i\right)\wedge *\left(dc^j - ldb^j\right) - \frac{1}{2}\mathcal{K}e^{2\phi}dl\wedge *dl \\
& -\frac{1}{2\mathcal{K}}e^{2\phi}\left(dh_1 - b^i d\rho_i\right)\wedge *\left(dh_1 - b^j d\rho_j\right) \\
& -\mathrm{e}^{4\phi}D\tilde{h}\wedge *D\tilde{h} \\
& +\frac{1}{2}Re\mathcal{M}_{AB}F^A\wedge F^B + \frac{1}{2}Im\mathcal{M}_{AB}F^A\wedge *F^B
\end{aligned}
\tag{2.88}
$$

with

$$
D\tilde{h} = dh_2 + ldh_1 + (c^i - lb^i)d\rho_i - \frac{1}{2}\mathcal{K}_{ijk}c^i c^j db^k. \tag{2.89}
$$

2.4 Mirror symmetry

Comparing the field content for the reduced type IIA and type IIB, comes the striking fact that the spectra are almost identical. The only difference lies in the number of vector and hypermultiplets. For type IIA, we found $h^{(1,1)}$ vector multiplets and $h^{(2,1)}+1$ hypermultiplets. For type IIB, it is exactly the other way around : $h^{(2,1)}$ vector multiplets and $h^{(1,1)}+1$ hypermultiplets. This strongly suggests that, defining the "mirror" manifold $\tilde{Y}$ of Y by

$$\tilde{Y} \quad : \quad \begin{cases} \tilde{h}^{(1,1)} &= h^{(2,1)} \\ \tilde{h}^{(2,1)} &= h^{(1,1)}, \end{cases} \tag{2.90}$$

type IIA compactified on Y would be identical to type IIB on $\tilde{Y}$. Moreover, when we compare (2.88) and (2.56), we can see that in the sector of the vectors the identification of the two matrices $\mathcal{N}$ and $\mathcal{M}$ is required. In order to prove that type IIA and type IIB are mirror image of each other, we need to show that this identification still holds for the hypermultiplets. Thus we expect to find, up to some field redefinitions, the metric for the hypermultiplets in type IIB

$$\begin{aligned} \tilde{h}_{uv}dq^u \wedge *dq^v &= g_{ij}dt^i \wedge *d\bar{t}^j + d\phi \wedge *d\phi \\ &+\frac{e^{4\phi}}{4}\left[da + (\tilde{\xi}_I d\xi^I - \xi^I d\tilde{\xi}_I)\right] \wedge^* \left[da + (\tilde{\xi}_J d\xi^J - \xi^J d\tilde{\xi}_J)\right] \\ &-\frac{e^{2\phi}}{2}\left(\mathrm{Im}\mathcal{N}^{-1}\right)^{IJ}\left[d\tilde{\xi}_I + \mathcal{N}_{IK}d\xi^K\right] \wedge^* \left[d\tilde{\xi}_J + \bar{\mathcal{N}}_{JL}d\xi^L\right]. \end{aligned} \tag{2.91}$$

Guided by the powers of the dilaton and the explicit expressions for the matrix $\mathcal{N}$ (B.85) and (B.86), we decompose the last line into

$$\left(\mathrm{Im}\mathcal{N}^{-1}\right)^{IJ}\left[d\tilde{\xi}_I + \mathrm{Re}\mathcal{N}_{IK}d\xi^K\right] \wedge^* \left[d\tilde{\xi}_J + \mathrm{Re}\mathcal{N}_{JL}d\xi^L\right] \tag{2.92}$$

$$+ \quad \mathrm{Im}\mathcal{N}_{IJ}d\xi^I \wedge *d\xi^J \tag{2.93}$$

and we expand

$$\mathrm{Im}\mathcal{N}_{IJ}d\xi^I \wedge *d\xi^J = -4\mathcal{K}g_{ij}\left(d\xi^i - b^i d\xi^0\right) \wedge * \left(d\xi^j - b^j d\xi^0\right) - \mathcal{K}d\xi^0 \wedge *d\xi^0 \tag{2.94}$$

This suggests to map

$$l \longleftrightarrow \xi^0 \tag{2.95}$$

$$lb^i - c^i \longleftrightarrow \xi^i. \tag{2.96}$$

In (2.92), the part of the component of $\left(\mathrm{Im}\mathcal{N}^{-1}\right)^{ij}$ which only contains the inverse metric g^{ij} reads

$$-\frac{1}{4\mathcal{K}}g^{ij}\left[d\tilde{\xi}_i + \mathrm{Re}\mathcal{N}_{iK}d\xi^K\right] \wedge^* \left[d\tilde{\xi}_j + \mathrm{Re}\mathcal{N}_{jL}d\xi^L\right]. \tag{2.97}$$

This leads to the identification

$$d\rho_i - \mathcal{K}_{ikl}c^k db^l \longleftrightarrow d\tilde{\xi}_i + \mathrm{Re}\mathcal{N}_{iK}d\xi^K \tag{2.98}$$

which yields, with (2.95) and (2.96),

$$\rho_i + \frac{1}{2}\mathcal{K}_{ikl}b^k b^l l - \mathcal{K}_{ikl}c^k b^l \longleftrightarrow \tilde{\xi}_i. \tag{2.99}$$

The rest of (2.92) can be written

$$-\frac{1}{\mathcal{K}}\Big[d\tilde{\xi}_0 + \mathrm{Re}\,\mathcal{N}_{0K}d\xi^K + b^i(d\tilde{\xi}_i + \mathrm{Re}\,\mathcal{N}_{iK}d\xi^K)\Big]\wedge{}^*\Big[d\tilde{\xi}_0 + \mathrm{Re}\,\mathcal{N}_{0L}d\xi^L + b^j(d\tilde{\xi}_j + \mathrm{Re}\,\mathcal{N}_{jL}d\xi^L)\Big],$$

where $\tilde{\xi}_0$ is identified as

$$-h_1 + \frac{1}{2}\mathcal{K}_{ikl}b^ib^kc^l - \frac{1}{6}\mathcal{K}_{ikl}b^ib^kb^l l \longleftrightarrow \tilde{\xi}_0. \tag{2.100}$$

The last of these redefinitions corresponds to the term

$$-\frac{e^{4\phi}}{4}\Big[da + (\tilde{\xi}_I d\xi^I - \xi^I d\tilde{\xi}_I)\Big]\wedge{}^*\Big[da + (\tilde{\xi}_J d\xi^J - \xi^J d\tilde{\xi}_J)\Big] \tag{2.101}$$

and gives us a

$$2h_2 + lh_1 + \rho_i(c^i - lb^i) \longleftrightarrow a. \tag{2.102}$$

Expressing the action (2.88) in this new set of fields, we finally find the mirror of (2.56)

$$\begin{aligned} S_{IIA} \;&=\; \int\Big[-\frac{1}{2}R^*\mathbf{1} - g_{ab}dz^a\wedge *d\bar{z}^b - \tilde{h}_{uv}dq^u\wedge *dq^v \\ &\quad +\frac{1}{2}\,\mathrm{Im}\,\mathcal{M}_{AB}F^A\wedge *F^B + \frac{1}{2}\,\mathrm{Re}\,\mathcal{M}_{AB}F^A\wedge F^B\Big] \end{aligned} \tag{2.103}$$

with $\tilde{h}_{uv}$ given in (2.91). The matrices $\mathcal{M}$ and $\mathcal{N}$ have the explicit expressions (B.84) and (B.111) which we recall here

$$\begin{aligned} \mathcal{N}_{IJ} &= \bar{\mathcal{F}}_{IJ} + \frac{2i}{X^M\mathrm{Im}\,\mathcal{F}_{MN}X^N}\mathrm{Im}\,\mathcal{F}_{IK}X^K\mathrm{Im}\,\mathcal{F}_{JL}X^L \\ \mathcal{M}_{AB} &= \bar{\mathcal{F}}_{AB} + \frac{2i}{z^E\mathrm{Im}\,\mathcal{F}_{EG}z^G}\mathrm{Im}\,\mathcal{F}_{AC}z^C\mathrm{Im}\,\mathcal{F}_{BD}z^D. \end{aligned} \tag{2.104}$$

in terms of prepotentials $\mathcal{F}$ depending holomorphically on the coordinates $z^A = (1, z^a)$ and $X^I = (1, t^i)$. The Kähler potentials can also be written using the prepotentials according to

$$\begin{aligned} \mathrm{e}^{-K_A} &= i\left(\bar{X}^I\mathcal{F}_I - X^I\bar{\mathcal{F}}_I\right) \\ \mathrm{e}^{-K_B} &= i\left(\bar{z}^A\mathcal{F}_A - z^A\bar{\mathcal{F}}_A\right), \end{aligned}$$

and are obviously mapped. To sum up, mirror symmetry states that type IIA on some Calabi-Yau manifold Y is the same as type IIB on the mirror manifold $\tilde{Y}$

$$IIA/Y \longleftrightarrow IIB/\tilde{Y} \tag{2.105}$$

with $\tilde{Y}$ obtained from Y by exchanging the even and odd cohomology classes following (2.90). This equivalence can be checked at the level of the supergravity by performing KK expansions of the type IIA and type IIB actions in 10 dimensions. The resulting actions are equivalent once the gauge coupling matrices $\mathcal{N}$ and $\mathcal{M}$ are mapped.

Chapter 3

Generalized Calabi-Yau compactifications

Let us review the main results of last chapter. In ten space-time dimensions there exist two inequivalent $N = 2$ supergravities denoted type IIA and type IIB. Both theories have the maximal amount of 32 local supersymmetries but they differ in their field content [1–3]. Phenomenologically interesting backgrounds correspond to compactifications on $\mathbb{R}^{1,3} \times Y$ where $\mathbb{R}^{1,3}$ is the four-dimensional Minkowski space while Y is a compact six-dimensional Euclidean manifold. The amount of supersymmetry which is left unbroken by the background depends on the holonomy group. The maximal holonomy group for a metric-compatible connection is $SO(6)$. It breaks all 32 supercharges whereas only some of the supercharges are broken by any of its subgroups. Calabi–Yau threefolds are a particularly interesting class of compactification manifolds as their holonomy group is $SU(3)$ and as a consequence they preserve only eight supercharges (2.14).

In a KK compactification on a Calabi–Yau threefold, the light modes of the effective theory are the coefficients of an expansion on solutions to Laplace equation with zero mass on Y. Such harmonic forms are the non-trivial elements of the cohomology groups $H^{(p,q)}(Y)$. The interactions of the light modes are captured by a low-energy effective Lagrangian $\mathcal{L}_{\text{eff}}$ which can be computed via a KK

reduction of the ten-dimensional Lagrangian. The resulting theory is a four-dimensional $N = 2$ supergravity coupled to vector- and hypermultiplets [15–17, 23–27]. Mirror symmetry relates the effective theories of type IIA and type IIB in 4 dimensions. Type IIA compactified on Y is equivalent to type IIB compactified on the mirror manifold $\tilde{Y}$ [28] defined by the exchange of the even and odd cohomology groups according to $H^{(1,1)}(Y) \leftrightarrow H^{(2,1)}(\tilde{Y})$ and vice-versa.

From the phenomenology point of view, one drawback of Calabi-Yau compactifications is the absence of a scalar potential lifting the vacuum degeneracy. One possible way to obtain a scalar potential is to include background fluxes. Type II supergravities contain several kinds of $(p-1)$-forms C_{p-1} with p-form field strengths $F_p = dC_{p-1}$. When the exterior derivative d is applied to such a form, expanded according to (2.10), it only acts on the space-time coefficients, due to harmonicity of the internal forms. Hence it is impossible to have a term with a space-time 0-form coefficient in the expansion of a field strength. However, remembering that a harmonic form is *locally* exact, one can consider a term of the form

$$F_p = e_i \omega_p^i \ , \tag{3.1}$$

where $\omega^i = d\chi^i$ is only true locally. Since we want F_p to be the exterior derivative of some form, e_i must be a constant, called a background flux. The name "background" comes from the fact

that such a term gives a background value to the field strength[1]

$$\int_{\gamma_p^i} F_p = e_i \ . \tag{3.2}$$

where γ_p^i is a p-cycle in Y Poincaré-dual to ω_p^i. For consistency it should be required that χ^i never appear explicitly in the action. This is easily achieved if the form C_{p-1} only participate to the action through its field strength F_p. We will always consider this case in the following.

Recently generalized Calabi–Yau compactifications of type II string theories have been considered where background fluxes for the field strengths F_p are turned on [6, 7, 29–36].

Due to a Dirac condition, the fluxes e_i are quantized in string theory. They are thus integers and their number is given by the Betti number h^p. However, in the supergravity approximation the fluxes can be considered as continuous parameters which represent a small perturbation of the original Calabi-Yau compactification. The light modes are no longer massless but acquire masses depending continuously on the fluxes. Nevertheless their induced masses are much smaller than the ones of the heavy KK states of order the compactification scale. The field content is consequently unchanged, and the interactions of the light modes continue to be captured by an effective Lagrangian $\mathcal{L}_{\text{eff}}$ which describes the dynamics of the fluctuations around the background values of the theory in the absence of fluxes. The fluxes appear as gauge or mass parameters and deform the original supergravity into a gauged or massive supergravity. They introduce a non-trivial potential for some of the massless fields and spontaneously break (part of) the supersymmetry.

$\mathcal{L}_{\text{eff}}$ has been computed in various situations. In refs. [7, 30, 31, 35] type IIB compactified on Calabi-Yau threefolds $\tilde{Y}$ in the presence of RR-three-form flux F_3 and NS-three-form flux H_3 was derived. In refs. [7, 29, 34] type IIA compactified on the mirror manifold Y with RR-fluxes F_0, F_2, F_4 and F_6 present was considered. The resulting low-energy effective action was equivalent to the type IIB action on the mirror manifold $\tilde{Y}$ with F_3 non-zero, but $H_3 = 0$ [7]. As expected given the matching of odd and even cohomologies on mirror pairs, the type IIB RR-fluxes F_3 in the third cohomology group $H^3(\tilde{Y})$ are mapped to the type IIA RR-fluxes in the even cohomology groups $H^0(Y)$, $H^2(Y)$, $H^4(Y)$ and $H^6(Y)$ [37, 38].

However, for non-vanishing NS-fluxes the situation is less clear as no obvious mirror symmetric compactification is known. In both type IIA and type IIB on Y an NS three-form H_3 exists which can give a non-trivial NS-flux in $H^3(Y)$. However, in neither case is there an NS form field which can give fluxes in the mirror symmetric even cohomologies $H^0(Y)$, $H^2(Y)$, $H^4(Y)$ and $H^6(Y)$. Vafa [8] suggested that the mirror symmetric configuration is related to compactifying on a manifold $\hat{Y}$ which is not complex but only admits a non integrable almost complex structure. The purpose of this chapter is to make this proposal more precise. As a first step we demand that the $D = 4$ effective action continues to have $N = 2$ supersymmetry, that is, eight local supersymmetries. According to (2.12), this implies that there is a single globally defined spinor η on $\hat{Y}$ so that each of the $D = 10$ supersymmetry parameters gives a single local four-dimensional supersymmetry. As a result, the structure group has to reduce from $SO(6)$ to $SU(3)$ or one of its subgroups. If we further demand that the two $D = 4$ supersymmetries are unbroken in a Minkowskian ground state η has to be covariantly constant with respect to the Levi-Civita connection ∇, see (2.13), or equivalently the holonomy group has to be $SU(3)$. This second requirement uniquely singles out Calabi–Yau threefolds as the correct compactification manifolds, see section 2.1.3. However, in this chapter we relax this second condition and only insist that a globally defined $SU(3)$-invariant spinor exists. Manifolds with this property have been discussed in the mathematics and physics literature and are known as manifolds with $SU(3)$ structure (see, for example, refs. [39–55]). They admit an almost complex structure J, a metric g which is hermitian with respect to J and a unique $(3,0)$-form Ω. Generically, since η is no longer covariantly constant, the Levi-Civita connection now fails to have $SU(3)$-holonomy.

[1] This is analogous to Gauss's theorem in electromagnetism $\oint \vec{E} \cdot d\vec{S} = Q$ where Q is the charge. Since the fluxes indeed parameterize some gaugings, they are usually called electric and magnetic fluxes.

However one can always write $\nabla\eta$ in terms of a three-index tensor, T^0, contracted with gamma matrices, acting on η. In the same way ∇J and $\nabla\Omega$ can be also written in terms of contractions of T^0 with J and Ω respectively. This tensor T^0, known as the intrinsic torsion, is thus a measure of the obstruction to having $SU(3)$ holonomy.

Different classes of manifolds with $SU(3)$ structure exist and they are classified by the different elements in the decomposition of the intrinsic torsion into irreducible $SU(3)$ representations. We will mostly consider the slightly non-generic situation where only "electric" flux is present. In this case, we find that mirror symmetry restricts us to a particular class of manifolds with $SU(3)$ structure called *half-flat* manifolds [44].[2] They are neither complex, nor Kähler, nor Ricci-flat but they are characterized by the conditions

$$d\Omega^- = 0 = d(J \wedge J) , \tag{3.3}$$

where Ω^- is the imaginary part of the $(3,0)$-form. On the other hand the real part of Ω is not closed and plays precisely the role of an NS four-form $d\Omega^+ \sim F_4^{NS}$ corresponding to fluxes along $H^4(Y)$ [8]. Thus the 'missing' NS-fluxes are purely geometrical and arise directly from the change in the compactification geometry.

This chapter is organized as follows. In section 3.1 we recall in more details mirror symmetry in Calabi-Yau compactifications with RR-flux. In section 3.2.1 we discuss properties of manifolds with $SU(3)$ structure and the way they realize supersymmetry in the effective action. These manifolds are classified in terms of irreducible representations of the structure group $SU(3)$ and in section 3.2.2 we argue that the class of half-flat manifolds are likely to be the mirror geometry of Calabi-Yau manifolds with electric NS-fluxes.

In section 3.3 we perform the KK-reduction of type IIA compactified on $\hat{Y}$, derive the low energy effective action and show that it is mirror symmetric to type IIB compactified on threefolds Y with non-trivial electric NS-flux H_3. The effect of the altered geometry is as expected. It turns an ordinary supergravity into a gauged supergravity in that scalar fields become charged and a potential is induced. This potential receives contributions from different terms in the ten-dimensional effective action, one of which arises from the non-vanishing Ricci-scalar. This contribution is crucial to obtain the exact mirror symmetric form of the potential. Of course, if $\hat{Y}$ is to be the mirror image of a Calabi-Yau manifold when NS fluxes are present, this relation should not depend on which theory one considers. Thus type IIB compactified on $\hat{Y}$ should also be equivalent to type IIA compactified on a Calabi-Yau manifold with NS fluxes. This is showed in section 3.4. Section 3.5 contains our conclusions. Calabi-Yau compactifications of type II theories with NS form-fluxes are reviewed in appendix C. Some technical details about G-structure are gathered in appendix

D while in appendix E we compute the Ricci-scalar for half-flat manifolds.

3.1 Mirror symmetry in CY compactifications with fluxes

Let us begin by reviewing mirror symmetry for Calabi–Yau compactifications with non-trivial RR fluxes. Consider first type IIB. The only allowed RR flux on the internal Calabi–Yau manifold $\tilde{Y}$ is the three-form $F_3 = dC_2$. The flux F_3 then defines $2(h^{(1,2)} + 1)$ flux parameters $(\tilde{e}_A, \tilde{m}^A)$ according to

$$F_3 = dC_2 + \tilde{m}^A \alpha_A - \tilde{e}_A \beta^A \ . \tag{3.4}$$

The effective action of this compactification is worked out in refs. [7,30,31,35]. A KK reduction is performed on the original Calabi-Yau geometry with the non-vanishing fluxes taken into account. This leads to a potential which induces small masses for some of the scalar fields and spontaneously breaks supersymmetry.

[2] Manifolds with torsion have also been considered in refs. [42,45–54,56]. However, in these papers the torsion is usually chosen to be completely antisymmetric in its indices or in other words it is a three-form. This turns out to be a different condition on the torsion and these manifolds are not half-flat.

It was shown in [7] that this IIB effective action is manifestly mirror symmetric to the one arising from the compactification of massive type IIA supergravity [57] on Y with RR-fluxes turned on in the even cohomology of Y. More precisely, in IIA compactifications the RR two-form field strength F_2 can have non-trivial flux in $H^2(Y)$ while the four-form field strength F_4 has fluxes in $H^4(Y)$. Then there are $2h^{(1,1)}$ IIA RR-flux parameters given by

$$F_2 = dA_1 + m^i \omega_i \ , \qquad F_4 = dC_3 - A_1 \wedge H_3 + e_i \tilde{\omega}^i \ . \tag{3.5}$$

In addition there are the two extra parameters m^0 and e_0, where e_0 is the dual of the space-time part of the four-form $F_{4\,\mu\nu\rho\sigma}$ and m^0 is the mass parameter of the original ten-dimensional massive type IIA theory [7]. Altogether there are $2(h^{(1,1)}+1)$ real RR-flux parameters $(e_I, m^J), I, J = 0, 1, \ldots, h^{(1,1)}$ which precisely map to the $2(h^{(1,2)}+1)$ type IIB RR-flux parameters under mirror symmetry. This is confirmed by an explicit KK-reduction of the respective effective actions and one finds [7][3]

$$\mathcal{L}^{(IIA)}(Y, e_I, m^J) \ \equiv \ \mathcal{L}^{(IIB)}(\tilde{Y}, \tilde{e}_A, \tilde{m}^B) \ . \tag{3.6}$$

We expect that mirror symmetry continues to hold when one considers fluxes in the NS-sector. However, in this case, the situation is more complicated. In both type IIA and type IIB there is a NS two-form B_2 with a three-form field strength H_3, so one can consider fluxes in $H^3(Y)$ in IIA and $H^3(\tilde{Y})$ in IIB. However, these are clearly not mirror symmetric since mirror symmetry exchanges the even and odd cohomologies. One appears to be missing $2(h^{(1,1)} + 1)$ NS-fluxes, lying along the even cohomology of Y and $\tilde{Y}$, respectively. Since the NS fields include only the metric, dilaton and two-form B_2, there is no candidate NS even-degree form-field strength to provide the missing fluxes. Instead, they must be generated by the metric and the dilaton. Thus we are led to consider compactifications on a generalized class of manifolds $\hat{Y}$ with a metric which is no longer Calabi–Yau, and perhaps a non-trivial dilaton in order to find a mirror-symmetric effective action. This necessity was anticipated by Vafa in ref. [8].

We now turn to what characterizes this generalized class of compactifications on $\hat{Y}$. We choose to first present the IIA compactification on a half-flat manifold $\hat{Y}$ compared to the IIB compactification on a Calabi-Yau manifold Y with NS flux H_3. Since the NS sectors are identical this should be, of course, equivalent to the problem with the roles of IIA and IIB reversed. This one will be addressed in section 3.4.

3.2 Half-flat spaces as mirror manifolds

3.2.1 Supersymmetry and manifolds with $SU(3)$-structure

The low-energy effective action arising from IIB compactifications with non-trivial H_3-flux describes a massive deformation of an $N = 2$ supergravity [7, 30, 31, 35]. Compactification on the conjectured generalized mirror IIA manifold $\hat{Y}$ should lead to the same effective action. Thus the first constraint on $\hat{Y}$ is that the resulting low-energy theory preserves $N = 2$ supersymmetry.

Let us first briefly review how supersymmetry is realized in the conventional Calabi–Yau compactification. Ten-dimensional type IIA supergravity has two supersymmetry parameters $\epsilon^{\pm}$ of opposite chirality each transforming in a real 16-dimensional spinor representation of the Lorentz group $Spin(1, 9)$. In particular, the variation of the two gravitini in type IIA is schematically given by [58]

$$\delta\psi_M^{\pm} = [\nabla_M + (\Gamma \cdot H_3)_M]\, \epsilon^{\pm} + [(\Gamma \cdot F_2)_M + (\Gamma \cdot F_4)_M]\, \epsilon^{\mp} + \ldots \ , \tag{3.7}$$

where the dots indicate further fermionic terms. Next one dimensionally reduces on a six-dimensional manifold Y and requires that the theory has a supersymmetric vacuum of the form

[3] For $m^I = 0$ one finds a standard $N = 2$ gauged supergravity with a potential for the moduli scalars of the vector multiplets. For $m^I \neq 0$ a non-standard supergravity occurs where the two-form B_2 becomes massive. For a more detailed discussion and a derivation of the effective action we refer the reader to ref. [7].

$\mathbb{R}^{1,3} \times Y$ with all other fields trivial. Following the discussion in section 2.1.3, this implies that there are particular spinors $\epsilon^{\pm}$ for which the gravitino variations (3.7) vanish. On $\mathbb{R}^{1,3} \times Y$ the Lorentz group $Spin(1,9)$ decomposes into $Spin(1,3) \times Spin(6)$ and we can write $\epsilon^{\pm} = \theta^{\pm} \otimes \eta$. In the supersymmetric vacuum, the vanishing of the gravitino variations imply the $\theta^{\pm}$ are constant and η is a solution of

$$\nabla_m \eta = 0 \ , \qquad m = 1, \ldots, 6 \ . \tag{3.8}$$

If this equation has a single solution, each $\epsilon^{\pm}$ gives a Killing spinor and we see that the background preserves $N = 2$ supersymmetry in four dimensions as required. Equivalently, if we compactify on Y, the low-energy effective action will have $N = 2$ supersymmetry and admits a flat supersymmetric ground state $\mathbb{R}^{1,3}$.

The condition (3.8) really splits into two parts: first the existence of a non-vanishing globally defined spinor η on Y and second that η is covariantly constant. The first condition implies the existence of two four-dimensional supersymmetry parameters and hence that the effective action has $N = 2$ supersymmetry. The second condition that η is covariantly constant implies that the effective action has a flat supersymmetric ground state.

The existence of η is equivalent to the statement that the structure group is reduced. To see what this means, recall that the structure group refers to the group of transformations of orthonormal frames over the manifold. Thus on a space-time of the form $\mathbb{R}^{1,3} \times Y$ the structure group reduces from $SO(1,9)$ to $SO(1,3) \times SO(6)$ and the spinor representation decomposes accordingly as $\mathbf{16} \to (\mathbf{2}, \mathbf{4}) + (\bar{\mathbf{2}}, \bar{\mathbf{4}})$. Suppose now that the structure group of Y reduces further to $SU(3) \subset SO(6) \cong SU(4)$. The $\mathbf{4}$ then decomposes as $\mathbf{3} + \mathbf{1}$ under the $SU(3)$ subgroup. An invariant spinor η in the singlet representation of $SU(3)$ thus depends trivially on the tangent space of Y and so is globally defined and non-vanishing. Conversely, the existence of such a globally defined spinor implies that the structure group of Y is $SU(3)$ (or a subgroup thereof). Mathematically, one says that Y has $SU(3)$-structure. In appendix D we review some of the properties of such manifolds from a more mathematical point of view and for a more detailed discussion we refer the reader to the mathematics literature [39–44]. Here we will concentrate on the physical implications.

The second condition that η is covariantly constant has well known consequences (as reviewed for instance in [59]). It is equivalent to the statements that the Levi–Civita connection has $SU(3)$ holonomy or similarly that Y is Calabi–Yau. It implies that an integrable complex structure exists and that the corresponding fundamental two-form J is closed. In addition, there is a unique closed holomorphic three-form Ω. Together these structure and integrability conditions imply that Calabi-Yau manifolds are complex, Ricci-flat and Kähler.

Symmetry with the low-energy IIB theory with H_3-flux implies that compactification on generalized mirror manifold $\hat{Y}$ still leads to an effective action that is $N = 2$ supersymmetric. However, the IIB theory with flux in general no longer has a flat-space ground state which preserves all supercharges [6,31,32]. From the above discussion, we see that this implies that we still have a globally defined non-vanishing spinor η, but we no longer require that η is covariantly constant, so $\nabla_m \eta \neq 0$. In other words, $\hat{Y}$ has $SU(3)$-structure but generically the Levi–Civita connection no longer has $SU(3)$-holonomy, so in general, $\hat{Y}$ is not Calabi–Yau. In particular, as discussed in appendix E, generic manifolds with $SU(3)$-structure are not Ricci-flat.

In analogy with Calabi-Yau manifolds let us first use the existence of the globally defined spinor η to define other invariant tensor fields.[4] Specifically, one has a fundamental two-form

$$J_{mn} = -i \eta^{\dagger} \Gamma_7 \Gamma_{mn} \eta \ , \tag{3.9}$$

and a three-form

$$\Omega = \Omega^{+} + i \Omega^{-} \ , \tag{3.10}$$

[4] For Calabi-Yau manifolds these constructions are reviewed, for example, in ref. [59]. For compactifications with torsion they are generalized in ref. [45, 46, 55, 60] and here we closely follow these references.

where

$$\Omega^+_{mnp} = -i\eta^\dagger\Gamma_{mnp}\eta \ , \qquad \Omega^-_{mnp} = -i\eta^\dagger\Gamma_7\Gamma_{mnp}\eta \ . \tag{3.11}$$

By applying Fierz identities one shows

$$\begin{aligned} J\wedge J\wedge J &= \frac{3\mathrm{i}}{4}\Omega\wedge\bar{\Omega} \ , \\ J\wedge\Omega &= 0 \ , \end{aligned} \tag{3.12}$$

exactly as for Calabi–Yau manifolds. Similarly, raising an index on J_{mn} and assuming a normalization $\eta^\dagger\eta = 1$, one finds

$$J_m{}^p J_p{}^n = -\delta_m{}^n \ , \qquad J_m{}^p J_n{}^r g_{pr} = g_{mn} \ , \tag{3.13}$$

by virtue of the Γ-matrix algebra (B.1). This implies that $J_m{}^p$ defines an almost complex structure such that the metric g_{mn} is Hermitian with respect to $J_m{}^p$. The existence of an almost complex structure is sufficient to define (p,q)-forms as we review in appendix D. In particular, one can see that Ω is a $(3,0)$-form.

Thus far we have used the existence of the $SU(3)$-invariant spinor η to construct J and Ω. One can equivalently characterize manifolds with $SU(3)$-structure by the existence of a globally defined, non-degenerate two-form J and a globally defined non-vanishing complex three-form Ω satisfying the conditions (3.12). Together these then define a metric [41, 61].

The key difference from the Calabi–Yau case is that a generic $\hat{Y}$ does not have $SU(3)$ holonomy since $\nabla_m\eta \neq 0$. Using (3.9) and (3.10) this immediately implies that also J and Ω are generically no longer covariantly constant $\nabla_m J_{np} \neq 0$, $\nabla_m\Omega_{npq} \neq 0$. In other words the deviation from being covariantly constant is a measure of the deviation from $SU(3)$ holonomy and thus a measure of the deviation from the Calabi–Yau condition. This can be made more explicit by using the fact that on $\hat{Y}$ there always exists another connection $\nabla^{(T)}$, which is metric compatible (implying $\nabla^{(T)}_m g_{np} = 0$), and which does satisfy $\nabla^{(T)}_m\eta = 0$ [40, 41]. The difference between any two metric-compatible connections is a tensor, known as the contorsion κ_{mnp}, and thus we have explicitly

$$\nabla^{(T)}_m\eta = \nabla_m\eta - \frac{1}{4}\kappa_{mnp}\Gamma^{np}\eta = 0 \ , \tag{3.14}$$

where Γ^{np} is the antisymmetrized product of Γ-matrices defined in appendix A.3 and κ_{mnp} takes values in $\Lambda^1\otimes\Lambda^2$ (Λ^p being the space of p-forms). We see that κ_{mnp} is the obstruction to η being covariantly constant with respect to the Levi-Civita connection and thus for non-vanishing κ the manifold $\hat{Y}$ can not be a Calabi-Yau manifold. Similarly, using (3.9), (3.10) and (3.14) one shows that also J and Ω are generically no longer covariantly constant but instead obey

$$\begin{aligned} \nabla^{(T)}_m J_{np} &= \nabla_m J_{np} - \kappa_{mn}{}^r J_{rp} - \kappa_{mp}{}^r J_{nr} = 0 \ , \\ \nabla^{(T)}_m\Omega_{nmp} &= \nabla_m\Omega_{npq} - \kappa_{mn}{}^r\Omega_{rpq} - \kappa_{mp}{}^r\Omega_{nrq} - \kappa_{mq}{}^r\Omega_{npr} = 0 \ , \end{aligned} \tag{3.15}$$

where again κ is measuring the obstruction to J and Ω being covariantly constant with respect to the Levi-Civita connection. We see that the connection $\nabla^{(T)}$ preserves the $SU(3)$ structure in that η or equivalently J and Ω are constant with respect to $\nabla^{(T)}$.

Let us now analyze the contorsion $\kappa\in\Lambda^1\otimes\Lambda^2$ in a little more detail. Recall that Λ^2 is isomorphic to the Lie algebra $so(6)$, which in turn decomposes into $su(3)$ and $su(3)^\perp$, with the latter defined by $su(3)\oplus su(3)^\perp \cong so(6)$. Thus the contorsion actually decomposes as $\kappa^{su(3)} + \kappa^0$ where $\kappa^{su(3)}\in\Lambda^1\otimes su(3)$ and $\kappa^0\in\Lambda^1\otimes su(3)^\perp$. Consider now the action of κ on the spinor η. Since η is an $SU(3)$ singlet, the action of $su(3)$ on η vanishes, and thus, from (3.14), we see that

$$\nabla_m\eta = \frac{1}{4}\kappa^0_{mnp}\Gamma^{np}\eta \ . \tag{3.16}$$

From (3.15), one finds that analogous expressions hold for $\nabla_m J_{np}$ and $\nabla_m\Omega_{npq}$. We see that the obstruction to having a covariantly constant spinor (or equivalently J and Ω) is actually measured by not the full contorsion κ but by the so-called "intrinsic contorsion" part κ^0. Eq. (3.16)

implies that κ^0 is independent of the choice of $\nabla^{(T)}$ satisfying (3.14), and thus is a property only of the $SU(3)$-structure. This fact is reviewed in more detail in appendix D.

Mathematically, it is sometimes more conventional to use the torsion T instead of the contorsion κ; the two are related via $T_{mnp} = \frac{1}{2}(\kappa_{mnp} - \kappa_{nmp})$ and T_{mnp} also satisfies (D.15). Similarly, one usually refers to the corresponding "intrinsic torsion" $T^0_{mnp} = \frac{1}{2}(\kappa^0_{mnp} - \kappa^0_{nmp})$ which also is an element of $\Lambda^1 \otimes su(3)^\perp$ and is in one-to-one correspondence with κ^0.[5] If κ^0 and hence T^0 vanishes, we say that the $SU(3)$ structure is torsion-free. This implies $\nabla_m \eta = 0$ and the manifold is Calabi–Yau.

Both κ^0 and T^0 can be decomposed in terms of irreducible $SU(3)$ representations and hence different $SU(3)$ structures can be characterized by the non-trivial $SU(3)$ representations T^0 carries. Adopting the notation used in [43, 44] we denote this decomposition by

$$T^0 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \ , \tag{3.17}$$

with the corresponding parts of T^0 labeled by T_i with $i = 1, \ldots, 5$ and where the representations corresponding to the different $\mathcal{W}_i$ are given in table 3.1.

component	interpretation	$SU(3)$-representation
$\mathcal{W}_1$	$J \wedge d\Omega$ or $\Omega \wedge dJ$	$\mathbf{1} \oplus \mathbf{1}$
$\mathcal{W}_2$	$(d\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
$\mathcal{W}_3$	$(dJ)_0^{2,1} + (dJ)_0^{1,2}$	$\mathbf{6} \oplus \bar{\mathbf{6}}$
$\mathcal{W}_4$	$J \wedge dJ$	$\mathbf{3} \oplus \bar{\mathbf{3}}$
$\mathcal{W}_5$	$d\Omega^{3,1}$	$\mathbf{3} \oplus \bar{\mathbf{3}}$

Table 3.1: The five classes of the intrinsic torsion of a space with $SU(3)$ structure.

The second column of table 3.1, gives an interpretation of each component of T^0 in terms of exterior derivatives of J and Ω. The superscripts refer to projecting onto a particular (p, q)-type, while the 0 subscript refers to the irreducible $SU(3)$ representation with any trace part proportional to J^n removed (see appendix D.0.3). This interpretation arises since, from (3.15), we have

$$\begin{aligned} dJ_{mnp} &= 6T^0_{[mn}{}^r J_{r|p]} \ , \\ d\Omega_{mnpq} &= 12T^0_{[mn}{}^r \Omega_{r|pq]} \ . \end{aligned} \tag{3.18}$$

These can then be inverted to give an expression for each component T_i of T^0 in terms of dJ and $d\Omega$. This is discussed in more detail from the point of view of $SU(3)$ representations in appendix D.0.3.

Manifolds with $SU(3)$ structure are in general not complex manifolds. An almost complex structure J (obeying (3.13)) necessarily exists but the integrability of J is determined by the vanishing of the Nijenhuis tensor $N_{mn}{}^p$. From its definition (D.4) we see that a covariantly constant J has a vanishing $N_{mn}{}^p$ and in this situation the manifold is complex and Kähler (as is the case for Calabi–Yau manifolds). However, for a generic J the Nijenhuis tensor does not vanish and is instead determined by the (con-) torsion using (D.4) and (3.15). Thus T^0 also is an obstruction to $\hat{Y}$ being a complex manifold. However, one can show [43, 44] that $N_{mn}{}^p$ does not depend on all torsion components but is determined entirely by the component of the torsion $T_{1\oplus 2} \in \mathcal{W}_1 \oplus \mathcal{W}_2$, through

$$N_{mn}{}^p = 8(T_{1\oplus 2})_{mn}{}^p \ . \tag{3.19}$$

Before we proceed let us summarize the story so far. The requirement of an $N = 2$ supersymmetric effective action led us to consider manifolds $\hat{Y}$ with $SU(3)$-structure. Such manifolds

[5] Note that our terminology is not very precise in that whenever we use the notion of torsion we in fact mean by this intrinsic torsion.

admit a globally defined $SU(3)$-invariant spinor η but the holonomy group of the Levi-Civita connection is no longer $SU(3)$. The deviation from $SU(3)$ holonomy is measured by the intrinsic (con-)torsion, and implies that generically the manifold is neither complex nor Kähler. However, the fundamental two-form J and the $(3,0)$-form Ω can still be defined; in fact their existence is equivalent to the requirement that $\hat{Y}$ has $SU(3)$-structure. Different classes of manifolds with $SU(3)$ structure are labeled by the $SU(3)$-representations in which the intrinsic torsion tensor resides. In terms of J and Ω this is measured by which components of the exterior derivatives dJ and $d\Omega$ are non-vanishing.

3.2.2 Half-flat manifolds

In general, we might expect that there are further restrictions on $\hat{Y}$ beyond the supersymmetry condition that it has $SU(3)$-structure. This would correspond to constraining the intrinsic torsion so that only certain components in table 3.1 are non-vanishing. We provide evidence for a particular set of constraints in the following subsections. Then, in section 3.3, we verify that these conditions do lead to the required mirror symmetric type IIA effective action.

Before doing so, however, let us consider two arguments suggesting how these constraints might appear. First, recall that the Kähler moduli on the Calabi–Yau manifold are paired with the B_2 moduli as an element $B_2 + \mathrm{i}J$ of $H^2(Y, \mathbb{C})$ where J is the Kähler form. Under mirror symmetry, these moduli map to the complex structure moduli of $\tilde{Y}$ which are encoded in the closed holomorphic $(3,0)$-form Ω. Turning on H_3 flux on the original Calabi–Yau manifold means that the real part of the complex Kähler form $B_2 + \mathrm{i}J$ is no longer closed. Under the mirror symmetry, this suggests that we now have a manifold $\hat{Y}$ where half of $\Omega = \Omega^+ + \mathrm{i}\Omega^-$, in particular Ω^+, is no longer closed. From table 3.1, we see that $d\Omega^{2,2}$ is related to the classes $\mathcal{W}_1$ and $\mathcal{W}_2$ which can be further decomposed into $\mathcal{W}_1^+ \oplus \mathcal{W}_1^-$ and $\mathcal{W}_2^+ \oplus \mathcal{W}_2^-$ giving

$$\begin{aligned} &T^+_{1\oplus 2} \text{ corresponding to } (d\Omega^+)^{2,2} \ , \\ &T^-_{1\oplus 2} \text{ corresponding to } (d\Omega^-)^{2,2} \ . \end{aligned} \tag{3.20}$$

Thus, the above result that only Ω^- remains closed suggests that,

$$T^-_{1\oplus 2} = 0 \ . \tag{3.21}$$

One might expect that it also implies that half of the $\mathcal{W}_5$ component vanishes. However, as discussed in [44], $(d\Omega^+)^{3,1}$ and $(d\Omega^-)^{3,1}$ are related, so, in fact, all of the component in $\mathcal{W}_5$ vanishes and we have in addition

$$T_5 = 0 \ . \tag{3.22}$$

The second argument comes from the fact that the intrinsic torsion T^0 should be such that it supplies the missing $2(h^{(1,1)}+1)$ NS-fluxes. In other words we need the new fluxes to be counted by the even cohomology of the original Calabi-Yau manifold Y. This implies that there should be some well-defined relation between $\hat{Y}$ and the Calabi-Yau manifold Y. We return to this relation in more detail in section 3.3.1 but here let us simply make the very naive assumption that we try to match the $SU(3)$ representations of the $H^{p,q}(Y)$ cohomology group with the $SU(3)$ representations of T^0. This suggests setting

$$T_4 = T_5 = 0 \ . \tag{3.23}$$

since the corresponding $H^{3,2}(Y)$ and $H^{3,1}(Y)$ groups vanish on Y. On the other hand $T_{1,2,3}$ can be non-zero as the corresponding cohomologies do exist on Y.

Taken together, these arguments suggest that the appropriate conditions might be

$$T^-_{1\oplus 2} = T_4 = T_5 = 0 \ . \tag{3.24}$$

This is in fact a known class of manifolds, denoted *half-flat* [44]. From table 3.1 it is easy to see that the necessary and sufficient conditions can be written as

$$\begin{aligned} d\Omega^- &= 0, \\ d\,(J \wedge J) &= 0. \end{aligned} \tag{3.25}$$

It will be useful in the following to have explicit expressions for the components of the intrinsic torsion T_1, T_2 and T_3 which are non-vanishing when the manifold is half-flat. From table 3.1 we recall that $T_{1\oplus 2}$ is in the same $SU(3)$ representation as a complex four-form $F^{(2,2)}$ of type $(2,2)$. Explicitly we have

$$(T_{1\oplus 2})_{mn}{}^p = F_{mnrs}\Omega^{rsp} + \bar{F}_{mnrs}\bar{\Omega}^{rsp} \ . \tag{3.26}$$

The half-flatness condition $T^-_{1\oplus 2} = 0$ just imposes that F is real ($F = \bar{F}$) so that

$$(T_{1\oplus 2})_{mn}{}^p \ = \ (T^+_{1\oplus 2})_{mn}{}^p \ = \ 2F^{(2,2)}_{mnrs}\,\Omega^{+rsp} \ , \tag{3.27}$$

where we have used (3.10). Explicitly, from the relations (3.18) one has that F is related to $d\Omega$ by[6]

$$F^{(2,2)}_{mnrs} \equiv \frac{1}{4||\Omega||^2}\,(d\Omega)^{2,2}_{mnrs} = \frac{1}{4||\Omega||^2}\,(d\Omega^+)^{2,2}_{mnrs} \ . \tag{3.28}$$

We will see in section 3.3 that this plays the role of the NS four-form which precisely complexifies the RR 4-form background flux in the low-energy effective action. This fact was anticipated in [8]. However, it will only generate the electric fluxes defined in (C.13), i.e. half of the missing NS-fluxes. As we said in the introduction, the treatment of the magnetic fluxes, corresponding to the NS two-form flux is more involved and will be discussed in a separate publication [62].

Similarly, we see from table 3.1 that the component T_3 of the torsion is in the same representation as a real traceless three-form $A_0^{(2,1)} + \bar{A}_0^{(1,2)}$ of type $(2,1)+(1,2)$ (see also appendix D.0.3). From (3.18) we see that this form is proportional to $(dJ)_0^{(2,0)}$. Explicitly we have

$$(T_3)_{mnp} = \frac{1}{4}\left(\delta_m^{m'}\delta_n^{n'} - J_m{}^{m'}J_n{}^{n'}\right)J_p{}^{p'}(dJ)_{m'n'p'} - 2F\,(\Omega^+)_{mnp} \ , \tag{3.29}$$

where by F we denoted the trace in complex indices $F_{\alpha\beta}{}^{\alpha\beta}$.

The remainder of the section focuses on providing evidence that equations (3.25) are indeed the correct conditions. Before doing so, recall that compactifications on manifolds with torsion have also been discussed in refs. [42, 45–54, 60]. The philosophy of these papers was slightly different in that they considered backgrounds where some of the p-form field strength were chosen non-zero and in order to satisfy $\delta\psi_m = 0$. Here instead we want the torsion to generate terms which mimic or rather are mirror symmetric to NS-flux backgrounds. As a consequence, one finds rather different conditions. Since in both cases one wants $N = 2$ supersymmetry in four dimensions, the class of manifolds discussed in [42, 45–52, 60] are also manifolds with $SU(3)$ structure. However, in these cases the torsion is a traceless real three-form. This implies $T \in \mathcal{W}_3 \oplus \mathcal{W}_4$ so that $T_1 = T_2 = T_5 = 0$. As a consequence the Nijenhuis tensor vanishes (since it depends only on $T_{1\oplus 2}$) and the manifolds are complex but not Kähler.

3.3 Type IIA on a half-flat manifold

Before we launch into the details of the dimensional reduction, recall that we are aiming at the derivation of a type IIA effective action which is mirror symmetric to the type IIB effective action obtained from compactifications on Calabi-Yau threefolds with (electric) NS 3-form flux H_3

[6]Note, that up to this point, the normalization $\eta^\dagger\eta = 1$ fixed the normalization of J and Ω. In the following it will be useful to allow an arbitrary normalization of Ω, thus we have included in this expression the general factor $||\Omega||^2 \equiv \frac{1}{3!}\Omega_{\alpha\beta\gamma}\bar{\Omega}^{\alpha\beta\gamma}$.

turned on. This effective theory is reviewed in section C.2 while the Calabi-Yau compactification of type IIA without fluxes is recalled in section 2.2.2. As we have stressed throughout, the central problem is that in IIA theory there is no NS form-field which can reproduce the NS-fluxes which are the mirrors of H_3 in the type IIB theory. Vafa suggested that the type IIA mirror symmetric configuration is a different geometry where the complex structure is no longer integrable [8], so that the compactification manifold $\hat{Y}$ is not Calabi–Yau. In the previous section we have already collected evidence that half-flat manifolds are promising candidates for $\hat{Y}$. The additional flux was characterized by the four-form $F^{(2,2)} \sim d\Omega^{2,2}$. The purpose of this section is to calculate the effective action, in an appropriate limit, for type IIA compactified on a half-flat $\hat{Y}$, and show that it is exactly equivalent to the known effective theory for the mirror type IIB compactification with electric flux.

The basic problem we are facing in this section is that so far we have no mathematical procedure for constructing a half-flat manifold $\hat{Y}$ from a given Calabi-Yau manifold Y. Instead we will give a set of rules for the structure of $\hat{Y}$ and the corresponding light spectrum by using physical considerations and in particular using mirror symmetry as a guiding principle. Specifically, we will write a set of two-, three- and four-forms on $\hat{Y}$ which are in some sense "almost harmonic". By expanding the IIA fields in these forms, we can then derive the four-dimensional effective action which is equivalent to the known mirror type IIB action.

3.3.1 The light spectrum and the moduli space of $\hat{Y}$

To derive the effective four-dimensional theory we first have to identify the light modes in the compactification such as the metric moduli. Unlike the case of a conventional reduction on a Calabi–Yau manifold, from the IIB calculation we know that the low-energy theory has a potential (C.34) and so not all the light fields are massless. In any dimensional reduction there is always an infinite tower of massive Kaluza–Klein states, thus we need some criterion for determining which modes we keep in the effective action.

Recall first how this worked in the type IIB case. One starts with a background Calabi–Yau manifold $\tilde{Y}$ and makes a perturbative expansion in the flux H_3. To linear order, H_3 only appears in its own equation of motion, while it appears quadratically in the other equations of motion, such as the Einstein and dilaton equations, so, heuristically,

$$\begin{aligned} \nabla^m H_{mnp} &= \dots \ , \\ R_{mn} &= H^2_{mn} + \dots \ . \end{aligned} \tag{3.30}$$

In the perturbation expansion we first solve the linear equation on $\tilde{Y}$ which implies that H_3 is harmonic. We then consider the quadratic backreaction on the geometry of $\tilde{Y}$ and the dilaton. The backreaction will be small provided H_3 is small compared to the curvature of the compactification, set by the inverse size of the Calabi–Yau manifold $1/\tilde{L}$. Recall, however, that in string theory the flux $\int_{\gamma_3} H$, where γ_3 is any three-cycle in $\tilde{Y}$ is quantized in units of α'. Consequently $H_3 \sim \alpha'/\tilde{L}^3$ and so for a small backreaction we require $H_3/\tilde{L}^{-1} \sim \alpha'/\tilde{L}^2$ to be small. In other words, we must be in the large volume limit where the Calabi–Yau manifold is much larger than the string length, which anyway is the region where supergravity is applicable. The Kaluza–Klein masses will be of order $1/\tilde{L}$. The mass correction due to H_3 is proportional to $\alpha'/\tilde{L}^3$ and so is comparatively small in the large volume limit. Thus in the dimensional reduction it is consistent to keep only the zero-modes on $\tilde{Y}$ which get small masses of order $\alpha'/\tilde{L}^3$ and to drop all the higher Kaluza–Klein modes with masses of order $1/\tilde{L}$.

We would like to make the same kind of expansion in IIA and think of the generalized mirror manifold $\hat{Y}$ as some small perturbation of the original Calabi–Yau Y mirror to $\tilde{Y}$ without flux. The problem we will face throughout this section is that we do not have, in general, an explicit construction of $\hat{Y}$ from Y. Thus we can only give general arguments about the meaning of such a limit. From the previous discussion we saw that it is the intrinsic torsion T^0 which measures the deviation of $\hat{Y}$ from a Calabi–Yau manifold. Thus we would like to think that in the limit where

T^0 is small $\hat{Y}$ approaches Y. The problem is that in general Y and $\hat{Y}$ have different topology. Thus, at the best, we can only expect that $\hat{Y}$ approaches Y locally in the limit of small intrinsic torsion. Put another way, the torsion, like H_3 is really "quantized". Consequently, it cannot really be put to zero, instead we can only try distorting the space to a limit where locally T^0 is small and then locally the manifold looks like Y.

This can be made slightly more formal in the following way. It is a general result [39] that the Riemann tensor of any manifold with $SU(n)$ structure has a decomposition as

$$R = R_{\text{CY}} + R_{\perp} \ , \tag{3.31}$$

where the tensor R_{CY} has the symmetry properties of the curvature tensor of a true Calabi–Yau manifold, so that, for instance the corresponding Ricci tensor vanishes. The orthogonal component $R_{\perp}$ is completely determined in terms of ∇T^0 and $(T^0)^2$. (Note that the corresponding decomposition of the Ricci scalar in the half-flat case is calculated explicitly in appendix E.) From this perspective, we can think of $R_{\perp}$ as a correction to the Einstein equation on a Calabi–Yau manifold, analogous to the H_3^2 correction in the IIB theory. In particular, if $\hat{Y}$ is to be locally like Y in the limit of small torsion, we require

$$R_{\text{CY}}(\hat{Y}) = R(Y) \ . \tag{3.32}$$

What, however, characterizes the limit where the intrinsic torsion is small? Unlike the IIB case the string scale does not appear in T^0. Typically both curvatures R_{CY} and $R_{\perp}$ are of order $1/\hat{L}^2$ where $\hat{L}$ is the size of $\hat{Y}$. Thus making $\hat{Y}$ large will not help us. Instead, we must consider some distortion of the manifold so that $R_{\perp} \ll R_{\text{CY}}$. What this distortion might be is suggested by mirror symmetry. We know that, without flux, a large radius $\tilde{Y}$ is mapped to Y with large complex structure. Thus we might expect that we are interested in the large complex structure limit of $\hat{Y}$.

In this limit, the conjecture is that $R_{\perp}(\hat{Y})$ becomes a small perturbation, with a mass scale much smaller than the Kaluza–Klein scale set by the average size of $\hat{Y}$. Thus, as in the IIB case, at least locally, the original zero modes on Y become approximate massless modes on $\hat{Y}$ gaining a small mass due to the non-trivial torsion. This suggests it is again consistent in this limit to consider a dimensional reduction keeping only the deformations of $\hat{Y}$ which correspond locally to zero modes of Y. This holds both for the ten-dimensional gauge potentials given in the case without flux in (2.39) and the deformations of the metric as in (B.20) and (B.29).

Having discussed the approximation, let us now turn to trying to identify this light spectrum more precisely and characterizing how the missing NS flux enters the problem. As discussed, it is the intrinsic torsion of $\hat{Y}$ which characterizes the deviation of $\hat{Y}$ from a Calabi–Yau manifold therefore we expect that this encodes the NS-flux parameters we are looking for. Mirror symmetry requires that these new NS-fluxes are counted by the even cohomology of the "limiting" Calabi-Yau manifold Y. As we saw above, in the case of half-flat manifolds this suggests that the real $(2,2)$-form $F \sim d\Omega$ on $\hat{Y}$, introduced in (3.27) and discussed by Vafa [8], can be viewed as specifying some "extra data" on Y which is a harmonic form $\zeta \in H^4(Y,\mathbb{R})$ (or equivalently $H^2(Y,\mathbb{R})$) measuring, at least part of, the missing NS flux.

As mentioned above, the problem is that we have no explicit construction of $\hat{Y}$ in terms of Y and some given flux ζ. Nonetheless, we expect, if mirror symmetry is to hold, that for each pair (Y,ζ) there is a unique half-flat manifold $\hat{Y}_\zeta$, so that there is a map

$$(Y,\zeta) \Leftrightarrow \hat{Y}_\zeta \ , \tag{3.33}$$

where, in the limit of small torsion (large complex structure), Y and $\hat{Y}_\zeta$ with the corresponding metrics are locally diffeomorphic. In fact, we can argue two more conditions. First, the identification (3.33) can be applied at each point in the moduli space of Y giving us, assuming uniqueness, a corresponding moduli space of $\hat{Y}_\zeta$. Furthermore, it can be checked , for the simple case of the torus, that the type IIB H_3-flux only effect the topology of $\hat{Y}$ in the sense that all

points in the moduli space of $\hat{Y}_\zeta$ for given flux had the same topology. Thus we see that, if mirror symmetry is to hold, the moduli space of metrics $\mathcal{M}(Y)$ and $\mathcal{M}(\hat{Y}_\zeta)$ of Y and $\hat{Y}$ are the same

$$\mathcal{M}(\hat{Y}_\zeta) = \mathcal{M}(Y) \qquad \text{for any given } \zeta \ , \tag{3.34}$$

where ζ only effects the topology of $\hat{Y}$. This gives the full moduli space of all $\hat{Y}_\zeta$ the structure of an infinite number of copies of $\mathcal{M}(Y)$ labeled by ζ.

More explicitly, the matching of moduli spaces means that for each (Ω, J) on Y, since $\hat{Y}_\zeta$ has $SU(3)$ structure, we have a unique corresponding (Ω, J) on $\hat{Y}$ and we must have a corresponding expansion in terms of a basis of forms on $\hat{Y}$

$$\begin{aligned} &\Omega = z^A\,\alpha_A - \mathcal{F}_A\,\beta^A \ , \qquad A = 0,1,\ldots,h^{(1,2)}(Y) \ , \\ &J = v^i\,\omega_i \ , \qquad i = 1,\ldots,h^{(1,1)}(Y) \ , \end{aligned} \tag{3.35}$$

where $z^A = (1, z^a)$ with $a = 1,\ldots,h^{(1,2)}(Y)$ and the z^a are the scalar fields corresponding to the deformations of the complex structure ($\mathcal{F}_A$ is defined in section B.5), while the v^i are the scalar fields corresponding to the Kähler deformations. The key point here is that although (α_A, β^A) form a basis for Ω and the ω_i form a basis for J they are not, in general, harmonic, and thus are not bases for $H^3(\hat{Y})$ and $H^{(1,1)}(\hat{Y})$. Locally, however, in the limit of small intrinsic torsion, they should coincide with the harmonic basis of $H^3(Y)$ and $H^{(1,1)}(Y)$ on Y. For $*J$ one has an analogous expansion in terms of four-forms on $\hat{Y}$ as in (B.26)

$$*J = 4\mathcal{K} g_{ij} \nu^i \tilde{\omega}^j \ , \qquad i = 1,\ldots,h^{(1,1)}(Y) \ , \tag{3.36}$$

where, again, there is no condition on $\tilde{\omega}^i$ being harmonic on $\hat{Y}$, but in the small torsion limit they again locally approach harmonic forms on Y.

The above expressions (3.35) and (3.36) have been written in terms of a prepotential $\mathcal{F}$ and a metric g_{ij} which is the metric on the moduli space just as for Y. If the low-energy effective action is to be mirror symmetric we necessarily have that the metrics on the moduli spaces $\mathcal{M}(\hat{Y}_\zeta)$ and $\mathcal{M}(Y)$ agree. This means that the corresponding kinetic terms in the low-energy effective action agree and implies the conditions

$$\int_{\hat{Y}} \omega_i \wedge \tilde{\omega}^j = \delta_i^j \ , \quad \int_{\hat{Y}} \alpha_A \wedge \beta^B = \delta_A^B \ , \quad \int_{\hat{Y}} \alpha_A \wedge \alpha_B = \int_{\hat{Y}} \beta^A \wedge \beta^B = 0 \ , \tag{3.37}$$

exactly as on Y in (B.25) and (B.28).

Now let us return to the flux and the restrictions implied by $\hat{Y}_\zeta$ being half-flat. Recall that we have argued that the four-form $F^{(2,2)} \sim (d\Omega)^{2,2}$ corresponds to a harmonic form $\zeta \in H^4(Y,\mathbb{Z})$ measuring the flux. Given the map between harmonic four-forms on Y and the basis $\tilde{\omega}^i$ introduced in (3.36), we are naturally led to rewrite (3.28) as

$$\begin{aligned} F^{(2,2)}_{mnpq} &\equiv \frac{1}{4||\Omega||^2}(d\Omega)^{2,2}_{mnpq} \\ &= \frac{1}{4||\Omega||^2}\, e_i\, \tilde{\omega}^i_{mnpq} \ , \qquad i = 1,\ldots,h^{(1,1)}(Y) \ , \end{aligned} \tag{3.38}$$

where the e_i are constants parameterizing the flux. Again, in the limit of small torsion, locally F is equivalent to a harmonic form on Y, namely ζ.

Inserting (3.35) into (3.38), we have

$$d\Omega = z^A d\alpha_A - \mathcal{F}_A d\beta^A = e_i \tilde{\omega}^i \ . \tag{3.39}$$

However, we argued that the flux only effects the topology of $\hat{Y}$ and does not depend on the point in moduli space. Thus, we require that this condition is satisfied independently of the choice of moduli $z^A = (1, z^a)$. This is only possible if we have

$$d\alpha_0 = e_i\, \tilde{\omega}^i \ , \qquad d\alpha_a = d\beta^A = 0 \ , \tag{3.40}$$

where α_0 is singled out since it is the only direction in Ω which is independent of z^a.[7] Furthermore, inserting (3.40) into (3.37) gives

$$e_i = \int \omega_i \wedge d\alpha_0 = -\int d\omega_i \wedge \alpha_0 \ . \tag{3.41}$$

Thus consistency requires

$$d\omega_i = e_i \beta^0 \ , \qquad d\tilde{\omega}^i = 0 \ , \tag{3.42}$$

where the second equation follows from (3.40).[8]

Eqs. (3.40) and (3.42) imply, just as we anticipated above that neither ω_i nor $\tilde{\omega}^i$ are harmonic. In particular, ω_i are no longer closed while the dual forms $\tilde{\omega}^i$ are no longer co-closed, since at least one linear combination $e_i\tilde{\omega}^i$ is exact. However, assuming for instance that e_1 is non-zero, the linear combinations

$$\omega'_i = \omega_i - \frac{e_i}{e_1}\omega_1 \ , \qquad i \neq 1 \ , \tag{3.43}$$

are harmonic in that they satisfy

$$d\omega'_i = d^\dagger \omega'_i = 0 \ , \tag{3.44}$$

where we used $d^\dagger\omega'_i = *d*\omega'_i \sim *d\tilde{\omega}'^i$. Thus there are still at least $h^{(1,1)}(Y) - 1$ harmonic forms ω'_i on $\hat{Y}$. The same argument can be repeated for H^3 where one finds $2h^{(1,2)}$ harmonic forms or in other words the dimension of H^3 has changed by two and we have together

$$h^{(2)}(\hat{Y}) = h^{(1,1)}(Y) - 1 \ , \qquad h^{(3)}(\hat{Y}) = h^{(3)}(Y) - 2 \ . \tag{3.45}$$

Physically this can be understood from the fact that some of the scalar fields gain a mass proportional to the flux parameters and no longer appear as zero modes of the compactification. Similarly, from mirror symmetry we do not expect the occurrence of new zero modes on $\hat{Y}$ as these would correspond to additional new massless fields in the effective action. This is also consistent with our expectation that $\hat{Y}$ is topologically different from Y which stresses the point that Y and $\hat{Y}$ can only be locally close to each other in the large complex structure limit.

Simply from the moduli space of $SU(3)$-structure of $\hat{Y}_\zeta$ and the relation (3.38) we have conjectured the existence of a set of forms on $\hat{Y}_\zeta$ satisfying the conditions (3.40) and (3.42) which essentially encode information about the topology of $\hat{Y}_\zeta$. We should now see if this is compatible with a half-flat structure. In particular we find, given (3.35),

$$\begin{aligned} dJ &= v^i e_i \beta^0 \ , \\ d\Omega &= e_i \tilde{\omega}^i \ . \end{aligned} \tag{3.46}$$

From the standard $SU(3)$ relation $J \wedge \Omega = 0$ we have that $\omega_i \wedge \alpha^A = \omega_i \wedge \beta^A = 0$ for all A and hence in particular $J \wedge dJ = 0$. Furthermore, since the e_i are real, $d\Omega^- = 0$. Thus we see that (3.40) and (3.42) are consistent with half-flat structure.[9] Furthermore, since dJ and $d\Omega$ completely determine the intrinsic torsion T^0, we see that all the components of T^0 are given in terms of the constants e_i without the need for any additional information.

Let us summarize. We proposed a set of rules for identifying the light modes for compactification on $\hat{Y}$ compatible with mirror symmetry and half-flatness. We first argued that in the limit of large complex structure the torsion of $\hat{Y}$ is small, and locally $\hat{Y}$ and Y are diffeomorphic, even though globally they have different topology. In this limit, the light spectrum corresponds

[7] Of course this corresponds to a specific choice of the symplectic basis of H^3. It is the same choice which is conventionally used in establishing the mirror map without fluxes.

[8] Strictly speaking also $d\omega_i = e_i\beta^0 + a^A\alpha_A + b_a\beta^a$ for some yet undetermined coefficients a^A, b_a solves (3.41). However by a similar argument as presented for the exterior derivative of ω_i one can see that any non-vanishing such coefficient will produce a nonzero derivative of α_a or/and β^A contradicting (3.40). From this one concludes that the only solution of (3.41) is (3.42).

[9] It would be interesting to calculate the moduli space of half-flat metrics on $\hat{Y}_\zeta$ directly and see that it agreed with, or at least had a subspace, of the form given by (3.35) and (3.36) together with (3.40) and (3.42).

to modes on $\hat{Y}$ which locally map to the zero modes of Y. This was made more precise by first noting that mirror symmetry implies a one-to-one correspondence between each pair of a Calabi–Yau manifold Y and flux $\zeta \in H^4(Y,\mathbb{Z})$ and a unique half-flat manifold $\hat{Y}_\zeta$. As a consequence the moduli space of half-flat metrics on $\hat{Y}_\zeta$ has to be identical with the moduli space of Calabi–Yau metrics on Y. This in turn implies that the metrics on these moduli spaces agree and a basis of forms for J and Ω exist on $\hat{Y}$ which coincides with the corresponding basis of harmonic forms on Y in the small torsion limit. Identifying the missing NS flux e_i as $F \sim d\Omega^{2,2} \sim e_i\tilde{\omega}^i$ led to a set of differential relations among this basis of forms in terms of the $h^{(1,1)}(Y)$ flux parameters e_i. We further showed that these relations are compatible with the conditions of half-flatness. As we will see more explicitly in the next section these forms give the correct basis for expanding the ten-dimensional fields on $\hat{Y}$ and obtaining a mirror symmetric effective action. We will find that the masses of the light modes are proportional to the fluxes and thus to the intrinsic torsion of $\hat{Y}$.

3.3.2 The effective action

In this section we present the derivation of the low energy effective action of type IIA supergravity compactified on the manifold $\hat{Y}$ described in sections 3.2.2 and 3.3.1. As argued in the previous section we insist on keeping the same light spectrum as for Calabi-Yau compactifications and therefore the KK-reduction is closely related to the reduction on Calabi-Yau manifolds which we recall in appendix 2.2.2. The difference is that the differential forms we expand in are no longer harmonic but instead obey

$$d\alpha_0 = e_i\tilde{\omega}^i\ , \qquad d\alpha_a = d\beta^A = 0\ , \qquad d\omega_i = e_i\beta^0\ , \qquad d\tilde{\omega}^i = 0\ . \tag{3.47}$$

However, we continue to demand that these forms have identical intersection numbers as on the Calabi-Yau or in other words obey unmodified (3.37). As we are going to see shortly the relations (3.47) are responsible for generating mass terms in the effective action consistent with the discussion in the previous section.[10]

Let us start from the type IIA action in $D = 10$ [3]

$$\begin{aligned} S &= \int e^{-2\hat{\phi}}\left(-\frac{1}{2}\hat{R}*\mathbf{1} + 2d\hat{\phi}\wedge *d\hat{\phi} - \frac{1}{4}\hat{H}_3\wedge *\hat{H}_3\right) \\ &\quad -\frac{1}{2}\int\left(\hat{F}_2\wedge *\hat{F}_2 + \hat{F}_4\wedge *\hat{F}_4\right) + \frac{1}{2}\int \hat{H}_3\wedge\hat{C}_3\wedge d\hat{C}_3\,, \end{aligned} \tag{3.48}$$

where the notations are explained in more detail in section 2.2.2. In the KK-reduction the ten-dimensional (hatted) fields are expanded in terms of the forms $\omega_i, \alpha_A, \beta^A$ introduced in (3.35)

$$\begin{aligned} \hat{\phi} &= \phi\ , \qquad \hat{A}_1 = A^0\ , \qquad \hat{B}_2 = B_2 + b^i\omega_i \\ \hat{C}_3 &= C_3 + A^i\wedge\omega_i + \xi^A\alpha_A + \tilde{\xi}_A\beta^A\ , \end{aligned} \tag{3.49}$$

where A^0, A^i are one-forms in $D = 4$ (they will generate $h^{(1,1)}$ vector multiplets and contribute the graviphoton to the gravitational multiplet) while $\xi^A, \tilde{\xi}_A, b^i$ are scalar fields in $D = 4$. The b^i combine with the Kähler deformations v^i of (3.35) to form the complex scalars $t^i = b^i + iv^i$ sitting in the $h^{(1,1)}$ vector multiplets. The $\xi^a, \tilde{\xi}_a$ together with the complex structure deformations z^a of (3.35) are members of $h^{(1,2)}$ hypermultiplets while $\xi^0, \tilde{\xi}_0$ together with the dilaton ϕ and B_2 form the tensor multiplet.

[10] Note that we are not expanding in the harmonic forms ω'_i defined in (3.43) but continue to use the non-harmonic ω_i. The reason is that in the ω_i-basis mirror symmetry will be manifest. An expansion in the ω'_i-basis merely corresponds to field redefinition in the effective action as they are just linear combinations of the ω_i.

The difference with Calabi-Yau compactifications results from the fact that the derivatives of $\hat{B}_2, \hat{C}_3$ in (3.49) are modified as a consequence of (3.47) and we find

$$\begin{aligned} d\hat{C}_3 &= dC_3 + (dA^i)\wedge\omega_i + (d\xi^A)\alpha_A + (d\tilde{\xi}_a)\beta^a + (d\tilde{\xi}_0 - e_iA^i)\beta^0 + \xi^0 e_i\tilde{\omega}^i \ , \\ d\hat{B}_2 &= dB_2 + (db^i)\omega_i + e_ib^i\beta^0 \ . \end{aligned} \tag{3.50}$$

We already see that the scalar $\tilde{\xi}_0$ becomes charged precisely due to (3.47) which is exactly what we expect from the type IIB action. However, on the type IIB side we have $(h^{(1,2)}+1)$ electric flux parameters while in (3.50) only $h^{(1,1)}$ fluxes e_i appear. The missing flux arises from the NS 3-form field strength $\hat{H}_3 = d\hat{B}_2$ in the direction of β^0. Turning on this additional NS flux amounts to a shift

$$\hat{H}_3 \to \hat{H}_3 + e_0\beta^0 \ , \tag{3.51}$$

where e_0 is the additional mass parameter. Using (3.50), (3.51) and (2.37) we see that the parameter e_0 introduced in this way naturally combines with the other fluxes e_i into

$$\begin{aligned} \hat{H}_3 &= dB_2 + db^i\omega_i + (e_ib^i + e_0)\beta^0 \ , \qquad (3.52)\\ \hat{F}_4 &= (dC_3 - A^0\wedge dB_2) + (dA^i - A^0db^i)\wedge\omega_i + D\xi^A\alpha_A + D\tilde{\xi}_A\beta^A + \xi^0 e_i\tilde{\omega}^i \ , \end{aligned}$$

where the covariant derivatives are given by

$$D\tilde{\xi}_0 = d\tilde{\xi}_0 - e_i(A^i + b^iA^0) - e_0A^0, \qquad D\xi^A = d\xi^A, \qquad D\tilde{\xi}_a = d\tilde{\xi}_a. \tag{3.53}$$

This formula is one of the major consequences of compactifying on $\hat{Y}$ (in particular of expanding the 10 dimensional fields in forms which are not harmonic) as one of the scalars, $\tilde{\xi}_0$, becomes charged.

From here on the compactification proceeds as in the massless case by inserting (3.52) into the action (3.48). Except for few differences which we point out, the calculation continues as in section 2.2.2 and we are not going to repeat this calculation here. Using (2.39), (3.50) and (3.52) one can see that the parameters e_0 and e_i give rise to new interactions coming from the topological term in (3.48)

$$\begin{aligned} \frac{1}{2}\int_{\hat{Y}} \hat{H}_3\wedge\hat{C}_3\wedge d\hat{C}_3 &= \frac{\xi^0}{2}dB_2\wedge A^ie_i - \frac{1}{2}dB_2\wedge\Big(\xi^0(d\tilde{\xi}_0 - e_iA^i) + \xi^a d\tilde{\xi}_a - \tilde{\xi}_A d\xi^A\Big) \\ &+\frac{\xi^0}{2}e_idb^i\wedge C_3 + \frac{1}{2}db^i\wedge A^j\wedge dA^k\mathcal{K}_{ijk} \\ &-\frac{\xi^0}{2}(e_ib^i + e_0)dC_3 - \frac{1}{2}(e_ib^i + e_0)\wedge C_3\wedge d\xi^0 \ , \end{aligned} \tag{3.54}$$

where $\mathcal{K}_{ijk}$ is defined in (B.19).

The 3-form C_3 in 4 dimensions carries no physical degrees of freedom. Nevertheless it can not be neglected as it may introduce a cosmological constant. Moreover when such a form interacts non-trivially with the other fields present in the theory as in (3.54) its dualization to a constant requires more care. Collecting all terms which contain C_3 we find

$$S_{C_3} = -\frac{\mathcal{K}}{2}(dC_3 - A^0\wedge dB_2)\wedge *(dC_3 - A^0\wedge dB_2) - \xi^0\,(e_ib^i + e_0)dC_3\,. \tag{3.55}$$

As shown in [63, 64] the proper way of performing this dualization is by adding a Lagrange multiplier λdC_3. The 3-form C_3 is dual to the constant λ which was shown to be mirror symmetric to a RR-flux in ref. [7] and consequently plays no role in the analysis here. Solving for dC_3, inserting the result back into (3.55) and in the end setting $\lambda = 0$ we obtain the action dual to (3.55)

$$S_{dual} = -\frac{(\xi^0)^2}{2\mathcal{K}}(e_ib^i + e_0)^2 - \xi^0\,(e_ib^i + e_0)A^0\wedge dB_2\,. \tag{3.56}$$

Finally, in order to obtain the usual $N = 2$ spectrum we dualize B_2 to a scalar field denoted by a. Due to the Green-Schwarz type interaction of B_2 (the first term in (3.54) and the second term in (3.56)) a is charged, but beside that the dualization proceeds as usually. Putting together all the pieces and after going to the Einstein frame one can write the compactified action in the standard $N = 2$ form

$$\begin{aligned} S_{IIA} \quad = \quad & \int \Big[-\frac{1}{2} R *\mathbf{1} - g_{ij} dt^i \wedge * d\bar{t}^j - h_{uv} Dq^u \wedge * Dq^v \\ & + \frac{1}{2} \operatorname{Im} \mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{2} \operatorname{Re} \mathcal{N}_{IJ} F^I \wedge F^J - V_{IIA} *\mathbf{1} \Big], \end{aligned} \tag{3.57}$$

where the gauge coupling matrix $\mathcal{N}_{IJ}$ and the metrics g_{ij}, h_{uv} are given in (B.85), (B.22) and (2.57) respectively. As explained in section 2.2.2 the gauge couplings can be properly identified after redefining the gauge fields $A^i \to A^i - b^i A^0$. We have also introduced the notation $I = (0, i) = 0, \ldots, h^{(1,1)}$ and so $A^I = (A^0, A^i)$. Among the covariant derivatives of the hypermultiplet scalars Dq^u the only non-trivial ones are[11]

$$Da = da - \xi^0 e_I A^I \, ; \qquad D\tilde{\xi}_0 = d\tilde{\xi}_0 + e_I A^I \, . \tag{3.58}$$

We see that two scalars are charged under a Peccei-Quinn symmetry as a consequence of the non-zero e_I.

Before discussing the potential V_{IIA} let us note that the action (3.57) already has the form expected from the mirror symmetric action given in section C.2. In particular the forms α_0 and β^0 in (3.47) single out the two scalars $\xi^0, \tilde{\xi}_0$ from the expansion of $\hat{C}_3$. ξ^0 maps under mirror symmetry to the RR scalar l which is already present in the $D = 10$ type IIB theory while $\tilde{\xi}_0$ maps to the charged RR scalar in type IIB. Moreover, using these identifications one observes that the gauging (3.58) is precisely what one obtains in the type IIB case with NS electric fluxes turned on (C.35).

Finally, we need to check that the potential from (3.57) coincides with the one obtained in the type IIB case (C.34). In the case of type IIA compactified on $\hat{Y}$ one can identify four distinct contributions to the potential: from the kinetic terms of $\hat{B}_2$ and $\hat{C}_3$, from the dualization of C_3 in 4 dimensions and from the Ricci scalar of $\hat{Y}$. We study these contributions in turn. We go directly to the four-dimensional Einstein frame which amounts to multiplying every term in the potential by a factor $e^{4\phi}$ coming from the rescaling of $\sqrt{-g}$, ϕ being the four-dimensional dilaton which is related to the ten-dimensional dilaton $\hat{\phi}$ by $e^{-2\phi} = e^{-2\hat{\phi}} \mathcal{K}$.

Using (3.52) we see that the kinetic term of $\hat{B}_2$ in (3.48) contributes to the potential

$$V_1 = \frac{e^{2\phi}}{4\mathcal{K}} (e_i b^i + e_0)^2 \int_{\hat{Y}} \beta^0 \wedge * \beta^0 = -\frac{e^{-2\phi}}{4\mathcal{K}} (e_i b^i + e_0)^2 \left[(\operatorname{Im} \mathcal{M})^{-1} \right]^{00} \, , \tag{3.59}$$

where the integral over $\hat{Y}$ was performed using (B.94), (B.112) and (B.26). Similarly, the kinetic term of $\hat{C}_3$ produces the following piece in the potential

$$V_2 = e^{4\phi} \, \frac{(\xi^0)^2}{8\mathcal{K}} \, e_i e_j g^{ij} \, , \tag{3.60}$$

where g^{ij} arises after integrating over $\hat{Y}$ using (B.26). Furthermore, (3.56) contributes

$$V_3 = e^{4\phi} \, \frac{(\xi^0)^2}{2\mathcal{K}} (e_i b^i + e_0)^2 \, . \tag{3.61}$$

Combining (3.59), (3.60) and (3.61) we arrive at

$$\begin{aligned} V_{IIA} \quad & = \quad V_g + V_1 + V_2 + V_3 \\ & = \quad V_g - \frac{e^{2\phi}}{4\mathcal{K}} (e_i b^i + e_0)^2 \left[(\operatorname{Im} \mathcal{M})^{-1} \right]^{00} - e^{4\phi} \, \frac{(\xi^0)^2}{2} e_I e_J \left[(\operatorname{Im} \mathcal{N})^{-1} \right]^{IJ} \, , \end{aligned} \tag{3.62}$$

[11] Up to a redefinition of the sign of the fluxes.

where we used the form of the matrix $(\mathrm{Im}\,\mathcal{N})^{-1}$ given in (B.86). V_g is a further contribution to the potential which arises from the Ricci scalar. Since $\hat{Y}$ is no longer Ricci-flat R contributes to the potential and in this way provides another sensitive test of the half-flat geometry.

In appendix E we show that for half-flat manifolds the Ricci scalar can be written in terms of the contorsion as

$$R = -\kappa_{mnp}\kappa^{npm} - \frac{1}{2}\epsilon^{mnpqrs}(\nabla_m\kappa_{npq} - \kappa_{mp}{}^{l}\kappa_{nlq})J_{rs} \ , \tag{3.63}$$

which, as expected, vanishes for $\kappa = 0$. In order to evaluate the above expression we first we have to give a prescription about how to compute $\nabla_m\kappa_{npq}$. Taking into account that at in the end the potential in the four-dimensional theory appears after integrating over the internal manifold $\hat{Y}$ we can integrate by parts and 'move' the covariant derivative to act on J. This in turn can be computed by using the fact that J is covariantly constant with respect to the connection with torsion (3.15). Replacing the contorsion κ from (D.18), going to complex indices and using the defining relations for the torsion (3.27), (3.29) and (3.38) one can find after some straightforward but tedious algebra the expression for the Ricci scalar. The calculation is presented in appendix E and here we only record the final result

$$R = -\frac{1}{8}\, e_i e_j g^{ij}\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00} \ . \tag{3.64}$$

Taking into account the factor $\frac{e^{-2\hat{\phi}}}{2}$ which multiplies the Ricci scalar in the 10 dimensional action (3.48) and the factor $e^{4\phi}$ coming from the four-dimensional Weyl rescaling one obtains the contribution to the potential coming from the gravity sector to be

$$V_g = -\frac{e^{2\phi}}{16\mathcal{K}}\, e_i e_j g^{ij}\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00} \ . \tag{3.65}$$

Inserted into (3.62) and using again (B.86) we can finally write the entire potential which appears in the compactification of type IIA supergravity on $\hat{Y}$

$$V_{IIA} = -\frac{e^{4\phi}}{2}\left((\xi^0)^2 - \frac{e^{-2\phi}}{2}\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00}\right) e_I e_J \left[(\mathrm{Im}\,\mathcal{N})^{-1}\right]^{IJ} . \tag{3.66}$$

In order to compare this potential to the one obtained in type IIB case (C.34) we should first see how the formula (3.66) changes under the mirror map. We know that under mirror symmetry the gauge coupling matrices $\mathcal{M}$ and $\mathcal{N}$ are mapped into one another. In particular this means that[12]

$$\left[(\mathrm{Im}\,\mathcal{M}_A)^{-1}\right]^{00} \leftrightarrow \left[(\mathrm{Im}\,\mathcal{N}_B)^{-1}\right]^{00} = -\frac{1}{\mathcal{K}_B} \ . \tag{3.67}$$

where we used the expression for $(Im\mathcal{N})^{-1}$ from (B.86). With this observation it can be easily seen that the type IIA potential (3.66) is precisely mapped into the type IIB one (C.34) provided one identifies the electric flux parameters $e_I \leftrightarrow \tilde{e}_A$ and the four-dimensional dilatons on the two sides.

To summarize the results obtained in this section, we have seen that the low energy effective action of type IIA theory compactified on $\hat{Y}$ is precisely the mirror of the effective action obtained in section C.2 for type IIB theory compactified on Y in the presence of NS electric fluxes. This is our final argument that the half-flat manifold $\hat{Y}$ is the right compactification manifold for obtaining the mirror partners of the NS electric fluxes of type IIB theory. In particular the interplay between the gravity and the matter sector which resulted in the potential (3.66) provided a highly nontrivial check on this assumption.

[12] In order to avoid confusions we have added the label A/B to specify the fact that the corresponding quantity appears in type IIA/IIB theory.

3.4 Type IIB on a half-flat manifold

In the previous section, Vafa's proposal that the mirror of the NS fluxes should come from the geometry of the internal manifold was made more concrete : it was conjectured that when NS fluxes are turned on in type IIB theory, mirror symmetry requires the presence of a new class of manifolds, known as half-flat manifolds with $SU(3)$ structure on type IIA side. The main argument supporting this proposal was provided by showing that the low-energy effective actions for the type IIB compactified on a Calabi-Yau three-fold in the presence of electric NS three-form flux and type IIA compactified on a half-flat space are equivalent. The purpose of this section is to test this conjecture in the reversed situation. We want to show that compactifying type IIB theory on half-flat manifolds produces an effective action which is mirror equivalent to type IIA theory compactified on Calabi-Yau three-folds with NS three-form flux turned on, whose action is reviewed in appendix C.1.

Following the previous sections, we will now perform the compactification of type IIB on a manifold $\hat{Y}$ obeying (3.40) and (3.42), which again will turn out to be responsible for generating mass terms and gaugings in the lower-dimensional action.

Let us start by shortly recording type IIB supergravity in ten dimensions. The NS-NS sector of the bosonic spectrum consists of the metric $\hat{g}_{MN}$, an antisymmetric tensor field $\hat{B}_2$ and the dilaton $\hat{\phi}$. In the RR sector one finds the 0-, 2-, and 4-form potentials $\hat{l},\ \hat{C}_2,\ \hat{A}_4$. The four-form potential satisfies a further constraint in that its field strength $\hat{F}_5$ is self-dual. The interactions of the above fields are described by the ten-dimensional action [3]

$$\begin{aligned} S^{(10)}_{IIB} &= \int e^{-2\hat{\phi}}\left(-\frac{1}{2}\hat{R}*\mathbf{1}+2d\hat{\phi}\wedge *d\hat{\phi}-\frac{1}{4}d\hat{B}_2\wedge *d\hat{B}_2\right) \\ &\quad -\frac{1}{2}\int\left(d\hat{l}\wedge *d\hat{l}+\hat{F}_3\wedge *\hat{F}_3+\frac{1}{2}\hat{F}_5\wedge *\hat{F}_5\right) \\ &\quad -\frac{1}{2}\int \hat{A}_4\wedge d\hat{B}_2\wedge d\hat{C}_2\ , \end{aligned} \tag{3.68}$$

where the field strengths $\hat{F}_3$ and $\hat{F}_5$ are defined as

$$\begin{aligned} \hat{F}_3 &= d\hat{C}_2-\hat{l}d\hat{B}_2\ , \\ \hat{F}_5 &= d\hat{A}_4-d\hat{B}_2\wedge\hat{C}_2\ . \end{aligned} \tag{3.69}$$

As it is well known the action (3.68) does not reproduce the correct dynamics of type IIB supergravity as the self-duality condition of $\hat{F}_5$ can not be derived from a variational principle. Rather this should be imposed by hand in order to obtain the correct equations of motion and we will come back to this constraint later as it plays a major role in the following analysis.

In order to compactify the action (3.68) on a half flat manifold we proceed as in section 3.3.2 and continue to expand the ten dimensional fields in the forms which appear in (3.40) and (3.42) even though they are not harmonic. The 4-dimensional spectrum is not modified by the introduction of the fluxes, and is still obtained by a regular KK expansion

$$\hat{B}_2 = B_2+b^i\wedge\omega_i\ ,\qquad i=1,\dots,h^{(1,1)}\ , \tag{3.70}$$
$$\hat{C}_2 = C_2+c^i\wedge\omega_i\ , \tag{3.71}$$
$$\hat{A}_4 = D^i_2\wedge\omega_i+\rho_i\wedge\tilde{\omega}^i+V^A\wedge\alpha_A-U_A\wedge\beta^A\ ,\qquad A=0,\dots,h^{(1,2)}\ ,$$

and thus one finds the two forms B_2, C_2, D^i_2, the vector fields V^A, U_A, and the scalars b^i, c^i, ρ_i. Additionally, from the metric fluctuations on the internal space one obtains the scalar fields z^a and v^i (3.35), which correspond to the Calabi-Yau complex structure and Kähler class deformations respectively. Due to the self-duality condition which one has to impose on $\hat{F}_5$, not all

the fields listed above describe physically independent degrees of freedom. Thus as four dimensional gauge fields one only encounters either V^A or U_A. In the same way, the scalars ρ_i and the two forms D_2^i are related by Hodge duality and one can eliminate either of the two in the four dimensional action. In the end one obtains an $N = 2$ supersymmetric spectrum consisting of a gravity multiplet $(g_{\mu\nu}, V^0)$, $h^{(2,1)}$ vector multiplets (V^a, z^a) and $4(h^{(1,1)} + 1)$ scalars $\phi,\ h_1,\ h_2,\ l,\ b^i,\ c^i,\ v^i,\ \rho^i$ which form $h^{(1,1)} + 1$ hypermultiplets.[13]

Up to this point everything looks like the ordinary Calabi-Yau compactification reviewed in section 2.3. The difference comes when one inserts the above expansion back into the action (3.68). Due to (3.40) and (3.42), the exterior derivatives of the fields (3.71) are going to differ from the standard case

$$\begin{aligned} d\hat{B}_2 &= dB_2 + db^i \wedge \omega_i + e_i b^i \beta^0 + e_0 \beta^0 \ , \\ d\hat{C}_2 &= dC_2 + dc^i \wedge \omega_i + e_i c^i \beta^0 \ , \\ d\hat{A}_4 &= dD_2^i \wedge \omega_i + e_i D_2^i \wedge \beta^0 + dV^A \wedge \alpha_A - dU_A \wedge \beta^A + (d\rho_i - e_i V^0) \wedge \tilde{\omega}^i \ . \end{aligned} \tag{3.72}$$

In analogy with type IIA case, we have also allowed for a normal H_3 flux proportional to β^0. This naturally combines with the other fluxes parameters e_i defined in (3.40) to provide all the $h^{(1,1)} + 1$ electric fluxes. With these expressions one can immediately write the field strengths F_3 and F_5 from (3.69)

$$\begin{aligned} \hat{F}_3 &= (dC_2 - l dB_2) + (dc^i - l db^i) \wedge \omega_i + e_i (c^i - l b^i) \beta^0 - l e_0 \beta^0 \ , \qquad (3.73) \\ \hat{F}_5 &= (dD_2^i - db^i \wedge C_2 - c^i dB_2) \wedge \omega_i + (D\rho_i - \mathcal{K}_{ijk} c^j db^k) \wedge \tilde{\omega}^i + F^A \wedge \alpha_A - \tilde{G}_A \wedge \beta^A \ , \end{aligned}$$

where we have defined

$$\begin{aligned} D\rho_i &= d\rho_i - e_i V^0 \ , \\ F^A &= dV^A \ , \quad G_A = dU_A \ , \\ \tilde{G}_0 &= G_0 - e_i (D_2^i - b^i C_2) + e_0 C_2 \ ; \quad \tilde{G}_a = G_a \ . \end{aligned} \tag{3.74}$$

In order to derive the lower-dimensional action we adopt the following strategy [35]. In the first stage we are going to ignore the self-duality condition which should be imposed on $\hat{F}_5$ and treat the fields coming from the expansion of $\hat{A}_4$ as independent. Thus, initially we naively insert the expansions (3.72) into (3.73) and perform the integrals over the internal space. To obtain the correct action we will further add suitable total derivative terms so that the self-duality conditions appear from a variational principle. At this point one can eliminate the redundant fields and in this way obtain the four-dimensional effective action and no other constraint has to be imposed. It can be checked that the result obtained in this way is compatible with the ten dimensional equations of motion.

Let us apply this procedure step by step. First one inserts the expansions (3.72) and (3.73) into the ten-dimensional action (3.68). The various terms of this action take the form

$$\begin{aligned} -\frac{1}{4}\int_Y d\hat{B}_2 \wedge * d\hat{B}_2 &= -\frac{\mathcal{K}}{4} dB_2 \wedge * dB_2 - \mathcal{K} g_{ij} db^i \wedge * db^j + \frac{1}{4}(e_i b^i + e_0)^2 \kappa_0 * \mathbf{1} \ , \\ -\frac{1}{2}\int_Y \hat{F}_3 \wedge * \hat{F}_3 &= -\frac{\mathcal{K}}{2} (dC_2 - l dB_2) \wedge * (dC_2 - l dB_2) \\ & \quad -2\mathcal{K} g_{ij} (dc^i - l db^i) \wedge * (dc^j - l db^j) + \frac{1}{2}\Big[e_i (c^i - l b^i) - l e_0\Big]^2 \kappa_0 * \mathbf{1} \ , \end{aligned}$$

[13] We have implicitly assumed that the two-forms C_2 and B_2 remain massless in four dimensions and they can be Hodge dualized to scalars which we have denoted h_1 and h_2 respectively.

$$\begin{aligned}
-\frac{1}{4}\int \hat{F}_5 \wedge *\hat{F}_5 &= +\frac{1}{4}\mathrm{Im}\,\mathcal{M}^{-1}\left(\tilde{G}-\mathcal{M}F\right)\wedge *\left(\tilde{G}-\bar{\mathcal{M}}F\right) \qquad (3.75)\\
&\quad -\mathcal{K}g_{ij}(dD_2^i - db^i\wedge C_2 - c^i dB_2)\wedge *(dD_2^j - db^j \wedge C_2 - c^j dB_2)\\
&\quad -\frac{1}{16\mathcal{K}}g^{ij}(D\rho_i - \mathcal{K}_{ilm}c^l db^m)\wedge *(D\rho_i - \mathcal{K}_{jnp}c^n db^p)\ ,\\
-\frac{1}{2}\int \hat{A}_4\wedge d\hat{B}_2 \wedge d\hat{C}_2 &= -\frac{1}{2}\mathcal{K}_{ijk}D_2^i\wedge db^j\wedge dc^k - \frac{1}{2}\rho_i\left(dB_2\wedge dc^i + db^i\wedge dC_2\right)\ ,\\
&\quad +\frac{1}{2}e_i V^0\wedge\left(c^i dB_2 - b^i dC_2\right) - \frac{1}{2}e_0 V^0\wedge dC_2\ .
\end{aligned}$$

In order to write the above formulae we have defined $\kappa_0 = \left(\mathrm{Im}\,\mathcal{M}^{-1}\right)^{00}$. In the gravitational sector, beyond the usual part containing the kinetic terms for the moduli of $\hat{Y}$ there will be a further contribution coming entirely from the internal manifold which is due to the fact that $\hat{Y}$ is not Ricci-flat and which will generate a piece of potential in four dimensions. The Ricci scalar for half-flat manifolds was computed in appendix E and here we will just record the effective potential generated in this way

$$V_g = -\frac{\kappa_0}{16\mathcal{K}}\mathrm{e}^{2\phi}e_i e_j g^{ij}\ . \qquad (3.76)$$

At this point we have to impose the self-duality condition for $\hat{F}_5$ which translates into the following constraints on the four dimensional fields

$$\begin{aligned}
dD_2^i - db^i\wedge C_2 - c^i dB_2 &= \frac{1}{4\mathcal{K}}g^{ij} * (D\rho_i - \mathcal{K}_{ijk}c^j db^k)\ ,\\
*\tilde{G}_A &= \mathrm{Re}\,\mathcal{M}_{AC} * F^C - \mathrm{Im}\,\mathcal{M}_{AC}F^C\ , \qquad (3.77)
\end{aligned}$$

with $D\rho_i$ and $\tilde{G}_A$ defined in (3.74). By adding the following total derivative term to the action

$$\begin{aligned}
\mathcal{L}_{\mathrm{td}} &= +\frac{1}{2}dD_2^i\wedge d\rho_i + \frac{1}{2}F^A\wedge G_A\\
&= +\frac{1}{2}dD_2^i\wedge D\rho_i + \frac{1}{2}F^A\wedge\tilde{G}_A - \frac{1}{2}(e_i b^i + e_0)F^0\wedge C_2 \qquad (3.78)
\end{aligned}$$

the constraints (3.77) can be found upon variation with respect to dD_2^i and G_A respectively. This allows us to eliminate the fields dD_2^i and G_A using their equations of motion and consequently the effective action obtained in this way describes the correct dynamics for the remaining fields which now do not have to satisfy any further constraint.

After the dualization of the 2-forms C_2 and B_2 to the scalars h_1 and h_2 one obtains the effective action for type IIB supergravity compactified to four dimensions on a half-flat manifold

$$\begin{aligned}
S_{IIB}^{(4)} = \int & -\frac{1}{2}R*\mathbf{1} - g_{ab}dz^a\wedge *d\bar{z}^b - g_{ij}dt^i\wedge *d\bar{t}^j - d\phi\wedge *d\phi\\
& -\frac{e^{2\phi}}{8\mathcal{K}}g^{-1\,ij}\left(D\rho_i - \mathcal{K}_{ikl}c^k db^l\right)\wedge *\left(D\rho_j - \mathcal{K}_{jmn}c^m db^n\right)\\
& -2\mathcal{K}e^{2\phi}g_{ij}\left(dc^i - l db^i\right)\wedge *\left(dc^j - l db^j\right) - \frac{1}{2}\mathcal{K}e^{2\phi}dl\wedge *dl \qquad (3.79)\\
& -\frac{1}{2\mathcal{K}}e^{2\phi}\left(dh_1 - b^i D\rho_i + e_0 V^0\right)\wedge *\left(dh_1 - b^j D\rho_j + e_0 V^0\right) - \mathrm{e}^{4\phi}D\tilde{h}\wedge *D\tilde{h}\\
& +\frac{1}{2}\mathrm{Re}\,\mathcal{M}_{AB}F^A\wedge F^B + \frac{1}{2}\mathrm{Im}\,\mathcal{M}_{AB}F^A\wedge *F^B - V_{IIB} * \mathbf{1}\ ,
\end{aligned}$$

where

$$D\tilde{h} = dh_2 + l dh_1 + (c^i - l b^i)D\rho_i + l e_0 V^0 - \frac{1}{2}\mathcal{K}_{ijk}c^i c^j db^k\ . \qquad (3.80)$$

Performing the field redefinitions [17]

$$
\begin{aligned}
a &= 2h_2 + lh_1 + \rho_i(c^i - lb^i) \ , \qquad \xi^0 = l \ , \qquad \xi^i = lb^i - c^i \ , \\
\tilde{\xi}_i &= \rho_i + \frac{l}{2}\mathcal{K}_{ijk}b^j b^k - \mathcal{K}_{ijk}b^j c^k \ , \qquad \tilde{\xi}_0 = -h_1 - \frac{l}{6}\mathcal{K}_{ijk}b^i b^j b^k + \frac{1}{2}\mathcal{K}_{ijk}b^i b^j c^k \ ,
\end{aligned} \tag{3.81}
$$

the metric for the hyperscalars takes the standard quaternionic form of [22] which is now exactly the mirror image of (C.12) with the gauge coupling matrices $\mathcal{N}$ and $\mathcal{M}$ exchanged as prescribed by the mirror map. Introducing the collective notation $q^u = (\phi, a, \xi^I, \tilde{\xi}_I)$ we can write the final form of the four dimensional action

$$
\begin{aligned}
S_{IIA} \;=\; & \int \Big[-\frac{1}{2}R^*\mathbf{1} - g_{ab}dz^a \wedge *d\bar{z}^b - \tilde{h}_{uv}Dq^u \wedge *Dq^v - V_{IIB} * \mathbf{1} \\
& + \frac{1}{2}\,\mathrm{Im}\,\mathcal{M}_{AB}F^A \wedge *F^B + \frac{1}{2}\,\mathrm{Re}\,\mathcal{M}_{AB}F^A \wedge F^B \Big] \ ,
\end{aligned} \tag{3.82}
$$

where the scalar potential has the form

$$
V_{IIB} = \frac{\kappa_0}{4}\mathrm{e}^{+2\phi}e_I e_J \left(\mathrm{Im}\,\mathcal{N}^{-1}\right)^{IJ} - \frac{\kappa_0}{2}\mathrm{e}^{4\phi}(e_I\xi^I)^2. \tag{3.83}
$$

The non-trivial covariant derivatives are

$$
D\tilde{\xi}_I = d\tilde{\xi}_I - e_I V^0 \ ; \qquad Da = da + e_I V^0 \xi^I \ , \tag{3.84}
$$

while all the other fields remain neutral.

This ends the derivation of the effective action of type IIB theory compactified to four dimensions on half-flat manifolds. One can immediately notice that the gaugings (3.84) are precisely the same as in the case of type IIA theory (C.6) and (C.9) when all the magnetic fluxes p^A are set to zero. It is not difficult to see that in this case also the potentials (3.83) and (C.11) coincide. For this one should just note that under mirror symmetry $\kappa_0 = (\mathrm{Im}\,\mathcal{M}_B^{-1})^{00}$ is mapped to $-\frac{1}{\mathcal{K}_A}$, $\mathcal{K}_A$ being the volume of the Calabi-Yau manifold on which type IIA is compactified.

3.5 Conclusions

In this chapter we proposed that type IIB (respectively IIA) compactified on a Calabi-Yau threefold $\tilde{Y}$ with electric NS three-form flux is mirror symmetric to type IIA (respectively IIB) compactified on a half-flat manifold $\hat{Y}$ with $SU(3)$ structure. The manifold $\hat{Y}$ is neither complex nor is it Ricci-flat. Nonetheless, though topologically distinct, it is closely related to the ordinary Calabi–Yau mirror partner Y of the original threefold $\tilde{Y}$. In particular, we argued that the moduli space of half-flat metrics on $\hat{Y}$ must be the same as the moduli space of Calabi–Yau metrics on Y. Furthermore, it is the topology of $\hat{Y}$ that encodes the even-dimensional NS-flux mirror to the original H_3-flux on $\tilde{Y}$.

We further strengthened this proposal by deriving the low-energy type IIA (IIB) effective action in the supergravity limit and showing that it is exactly equivalent to the appropriate type IIB (IIA) effective action. In particular, the resulting potential delicately depends on the non-vanishing Ricci scalar of the half-flat geometry and thus provided a highly non-trivial check on our proposal.

It is interesting to note that one particular NS flux e_0 played a special role in that it did not arise from the half-flat geometry but appeared as a NS three-form flux $H_3 \in H^{(3,0)}(\tilde{Y})$. In this context, it appears that mirror symmetry only acts on the 'interior' of the Hodge diamond in that it exchanges $H^{(1,1)} \leftrightarrow H^{(1,2)}$ but leaves $H^{(3,3)} \oplus H^{(0,0)}$ and $H^{(3,0)} \oplus H^{(0,3)}$ untouched. Put another way, it appears that it is the same single NS electric flux which is associated to both $H^{(3,0)} \oplus H^{(0,3)}$ and $H^{(3,3)} \oplus H^{(0,0)}$ on a given Calabi–Yau manifold.

We found that requirements of mirror symmetry provided a number of conjectures about the geometry of the half-flat manifold $\hat{Y}$. For instance the cohomology groups of $\hat{Y}$ shrink compared

to those of Y in that the Hodge numbers $h^{(1,1)}$ and $h^{(1,2)}$ are reduced by one. In addition, a non-standard KK reduction had to be performed in order to obtain masses for some of the scalar fields. This in turn led us to make a number of assumptions which need to be better understood from a mathematical point of view. One particular conjecture is the following. In general, the electric NS H_3-flux maps under mirror symmetry to some element $\zeta \in H^4(Y)$. Mirror symmetry would appear to imply that

> for all integer fluxes $\zeta \in H^4(Y)$ there should be a unique manifold $\hat{Y}_\zeta$ admitting a family of half-flat metrics such that the moduli space of such metrics $\mathcal{M}(Y_\zeta)$ is equal to the moduli space $\mathcal{M}(Y)$ of Calabi–Yau metrics on Y.

We note that it should be possible to determine this moduli space of half-flat geometries directly from its definition and without relying on the physical relation with Calabi-Yau threefold compactification.[14] Moreover, a more precise mathematical statement about the relationship between a given Calabi-Yau threefold Y and its 'cousin' half-flat geometry on $\hat{Y}$ should also be possible.

Finally, our analysis only treated electric NS fluxes. The discussion of the magnetic ones is technically more involved. Indeed, when type IIB is compactified on a Calabi-Yau manifold with magnetic NS form fluxes, a massive RR two-form appears which has no obvious counterpart on the type IIA side. In the second approach where type IIA is compactified on a Calabi-Yau manifold with magnetic NS three-form fluxes, no massive forms are present. Thus, it appears that in this picture it would be easier to look for the magnetic fluxes. However, in this case we encounter an other puzzle. Recall that when type IIA is compactified on a Calabi-Yau 3-fold with NS electric and magnetic fluxes (C.6), all the scalars ξ^A and $\tilde{\xi}_A$ are gauged, with respect to the same vector A^0. The corresponding vector in type IIB is V^0 which only appears in the expansion of the self-dual field strength $\hat{F}_5$. With a quick look at the action (2.58) one can immediately see that the scalars ξ^I, that are l and $lb^i - c^i$, whose kinetic term do not involve any $\hat{A}_4$ or $\hat{F}_5$, have no reason to be gauged under V^0. Thus it appears that some additional important ingredient still has to be found in order to reconcile electric and magnetic fluxes.

Nevertheless, we think that a few lines can be drawn. Recall that in the definition of the NS fluxes

$$H_3 \sim m^A \alpha_A - e_A \beta^A, \tag{3.1}$$

the electric and magnetic fluxes are treated on the same footing; they are coefficients of an expansion on a basis $(\alpha_A\, ,\, \beta^B)$ of harmonic forms on H^3. In type IIB, when only electric fluxes are turned on, the spectrum contains the usual massless RR 2-form C_2. However, when only magnetic fluxes are present, C_2 becomes massive, with a mass proportional to the fluxes (C.26). Since the notion of "electric" or "magnetic" is just a matter of choice of basis, how is it possible that magnetic and electric fluxes lead to so different results? Actually the difference only lies in the way one distributes the degrees of freedom. In [7], it was showed in a similar situation[15] that the massive 2-form is dual to a scalar and a massive vector. The scalar has the exact same couplings as would have the dual of the massless 2-form, and with the emergence of the extra vector comes a symmetry which allows to eliminate one vector. Thus the number of physical degrees of freedom is indeed unchanged. Following this idea, on can note that C_2 becomes massive when magnetic fluxes are present only because, in the process of discarding half of the fields due to self-duality of $\hat{F}_5$, the independent vectors in the expansion of $\hat{A}_4$ are chosen to be the V^A. If one chooses instead to keep the U_A, then C_2 is no longer massive, and one can check that the magnetic fluxes are mapped correctly, provided the rules for mapping are slightly modified.

[14] In this respect a generalization of ref. [65] might be useful.

[15] The massive form was the NS 2-form B_2 and the fluxes were coming from the RR sector, but the results can be easily extended to our case.

The problem arises when both kinds of fluxes are turned on. If one keeps V^A, then C_2 acquires a mass proportional to m^A, but if one keeps U_A, C_2 is also massive with a mass proportional to e_A. This means that the obstruction to having both fluxes at the same time seems to be related to a matter of dualization. Since the electric fluxes parameterize the failure of Ω^+ to be closed (3.28), the magnetic fluxes may naturally be involved in its failure to be co-closed. One may then want to impose

$$d^\dagger \Omega^+ = m^i \omega_i. \tag{3.2}$$

Recall that Ω^+ and Ω^- are related by $*\Omega^+ \sim \Omega^-$. This would suggest that magnetic fluxes require a further generalization of the half-flat geometry allowing the possibility $d\Omega^- \neq 0$.

Chapter 4

Equations of motion for Nicolai-Townsend multiplet

4.1 Introduction

The action for $N = 4$ supergravity theory containing an antisymmetric tensor was first given by Nicolai and Townsend [12] already in the early eighties. It was derived from the Lagrangian of [66] after a standard dualization of the pseudo-scalar. These theories are best understood as coming from compactification of $N = 1$ $d = 10$ pure supergravity on a six-dimensional torus T^6 [11]. A consistent truncation of type IIA supergravity in 10 dimensions is realized by turning off the RR fields A_1 and C_3 in the bosonic sector, and by keeping only half of the fermions, one gravitino and one dilatino [11, 67]. The resulting action describes the dynamics of the $N = 1$ gravity multiplet, composed of the metric $\hat{g}_{MN}$, the dilaton $\hat{\phi}$ and the antisymmetric NS tensor $\hat{B}_{MN}$ as bosonic fields, one gravitino and one dilatino as fermionic fields. With a simple counting of the bosonic degrees of freedom, one can deduce how the reduced fields arrange in $N = 4$ multiplets. The reduction of the metric leads to the metric in 4 dimensions $g_{\mu\nu}$, 6 vectors $A_{\mu m}$ and 21 scalars A_{mn}. The antisymmetric tensor gives one antisymmetric tensor $B_{\mu\nu}$, 6 vectors $B_{\mu m}$ and 15 scalars B_{mn}. In 4 dimensions, the $N = 4$ gravity multiplet contains the metric, 6 vectors, one scalar and one pseudo-scalar dual to an antisymmetric tensor. Substracting these degrees of freedom from the full spectrum, one is left with 6 vectors and 36 scalars, which lie in 6 vector multiplets[1].

More precisely, apart from the metric, the fields belonging to the gravity multiplet are the following : 6 vectors (graviphotons) $v_\mu{}^{\mathbf{u}}$, $\mathbf{u} = 1..6$, corresponding to a particular linear combination[2] of $A_{\mu m}$ and $B_{\mu m}$, the antisymmetric tensor and the dilaton. The superspace formulation of this multiplet, which we call the N-T multiplet in the following, encountered a number of problems identified in [68] and overcome in [69] by introducing external Chern-Simons forms for the graviphotons. Recently, a concise geometric formulation was given for this supergravity theory in central charge superspace [13].

The geometric approach adopted and described in detail in [13] was based on the superspace soldering mechanism involving gravity and 2–form geometries in central charge superspace [70]. This soldering procedure allowed to identify various gauge component fields of the one and the same multiplet in two distinct geometric structures: graviton, gravitini and graviphotons in the gravity sector and the antisymmetric tensor in the 2–form sector. Supersymmetry and central charge transformations of the component fields were deduced using the fact that in the geometric approach these transformations are identified on the same footing with general space-time coordinate transformations as superspace diffeomorphisms on the central charge superspace. Moreover, the presence of graviphoton Chern-Simons forms in the theory was interpreted as an

[1] A $N = 4$ vector multiplet contains one vector and 6 scalars as bosonic fields.

[2] The other independent linear combination of the vectors $A_{\mu m}$ and $B_{\mu m}$ corresponds to the vectors, which, together with the 36 scalars g_{mn} and B_{mn}, form the 6 vector multiplets.

intrinsic property of central charge superspace and a consequence of the superspace soldering mechanism.

The aim of the present work is to emphasize that the geometric description in [13] is on-shell, that is the constraints used to identify the component fields of the N-T supergravity multiplet imply also the equations of motion for these fields. Therefore, we begin with recalling briefly the basics of extended supergravity in superspace formalism. Then we identify the component fields [13] and specify the constraints we use. In section 3, we derive the equations of motion directly from constraints and Bianchi identities, without any knowledge about a Lagrangian. Finally, we compare these equations of motion with those found from the component Lagrangian given in the original article by Nicolai and Townsend [12].

4.2 Extended supergravities in superspace

In this section we briefly describe the geometry of extended superspace. We only give the notions that will be relevant to the remainder of this chapter. Very detailed accounts of superspace formalism can be found for example in [71] [72] for $N = 1$, and in [73] [74] for extended supergravities. We use the exact same conventions as in [72, 73]. The superspace is made of the space-time coordinates x^m, with additional fermionic coordinates $\theta^\alpha_{\rm A}, \theta^{\rm A}_{\dot\alpha}$ and bosonic central charges coordinates $z^{\mathbf{u}}$, where the index $_{\rm A}$ counts the number of supercharges and $\mathbf{u}$ the number of central charges. As usual, we define the vielbein one-form

$$E^{\mathcal{A}} \;=\; dz^{\mathcal{M}} E_{\mathcal{M}}{}^{\mathcal{A}} \tag{4.1}$$

where $z^{\mathcal{M}} = (x^m, \theta^\alpha_{\rm A}, \theta^{\rm A}_{\dot\alpha}, z^{\mathbf{u}})$ is the generalized coordinate on the superspace. The generic structure group is $SL(2,C) \times U(N)$, with connection $\Phi_{\mathcal{A}}{}^{\mathcal{B}}$. The covariant derivatives act on tensor fields in the following way

$$Du^{\mathcal{A}} \;=\; du^{\mathcal{A}} + u^{\mathcal{B}} \Phi_{\mathcal{B}}{}^{\mathcal{A}} \tag{4.2}$$

$$Dv_{\mathcal{A}} \;=\; dv_{\mathcal{A}} - (-)^{\deg(v)} \Phi_{\mathcal{A}}{}^{\mathcal{B}} v_{\mathcal{B}} \tag{4.3}$$

and lead to the algebra

$$(D_{\mathcal{C}}\, ,\, D_{\mathcal{B}})\, u^{\mathcal{A}} \;=\; -T_{\mathcal{C}\mathcal{B}}{}^{\mathcal{F}} D_{\mathcal{F}} u^{\mathcal{A}} + R_{\mathcal{C}\mathcal{B}\mathcal{F}}{}^{\mathcal{A}} u^{\mathcal{F}} \tag{4.4}$$

$$(D_{\mathcal{C}}\, ,\, D_{\mathcal{B}})\, v_{\mathcal{A}} \;=\; -T_{\mathcal{C}\mathcal{B}}{}^{\mathcal{F}} D_{\mathcal{F}} v_{\mathcal{A}} - R_{\mathcal{C}\mathcal{B}\mathcal{A}}{}^{\mathcal{F}} v_{\mathcal{F}} \tag{4.5}$$

where the torsion and the Riemann tensor are

$$T^{\mathcal{A}} \;=\; \mathrm{d}E^{\mathcal{A}} + E^{\mathcal{B}} \Phi_{\mathcal{B}}{}^{\mathcal{A}} \tag{4.6}$$

$$R_{\mathcal{A}}{}^{\mathcal{B}} \;=\; \mathrm{d}\Phi_{\mathcal{A}}{}^{\mathcal{B}} + \Phi_{\mathcal{A}}{}^{\mathcal{C}} \Phi_{\mathcal{C}}{}^{\mathcal{B}}. \tag{4.7}$$

These tensors are subject to consistency conditions expressed by the Bianchi Identities

$$DT^{\mathcal{A}} \;=\; E^{\mathcal{B}} R_{\mathcal{B}}{}^{\mathcal{A}} \tag{4.8}$$

$$DR_{\mathcal{B}}{}^{\mathcal{A}} \;=\; 0. \tag{4.9}$$

In the case of the superspace without central charges, Dragon's theorem [75] states that using (4.8), one can express all the components of the Riemann tensor in terms of the components

of the torsion. Moreover, once (4.8) is solved, (4.9) is identically satisfied. Under very mild assumptions [73] [74], this theorem can be extended to the superspace with central charges. This is why in our case we will only consider (4.8). In components it reads

$$\left({}_{\mathcal{DCB}}{}^{\mathcal{A}}\right)_T \quad : \quad E^{\mathcal{B}}E^{\mathcal{C}}E^{\mathcal{D}}\left(\mathcal{D}_{\mathcal{D}}T_{\mathcal{CB}}{}^{\mathcal{A}} + T_{\mathcal{DC}}{}^{\mathcal{F}}T_{\mathcal{FB}}{}^{\mathcal{A}} - R_{\mathcal{DCB}}{}^{\mathcal{A}}\right) = 0. \tag{4.10}$$

4.3 Identification of the fields

In this section we recall the essential results of [13] concerning the identification of the components of the N-T multiplet. Recall that in geometrical formulation of supergravity theories the basic dynamic variables are chosen to be the vielbein and the connection. Considering central charge superspace this framework provides a unified geometric identification of graviton, gravitini and graviphotons in the frame $E^{\mathcal{A}} = (E^a, E^{\alpha}_{\mathrm{A}}, E^{\mathrm{A}}_{\dot{\alpha}}, E^{\mathbf{u}})$.

$$E^a\| \;=\; dx^\mu e_\mu{}^a\,, \quad E^{\alpha}_{\mathrm{A}}\| \;=\; \frac{1}{2}\,dx^\mu \psi_{\mu}{}^{\alpha}_{\mathrm{A}}\,, \quad E^{\mathrm{A}}_{\dot{\alpha}}\| \;=\; \frac{1}{2}\,dx^\mu \bar{\psi}_{\mu}{}^{\mathrm{A}}_{\dot{\alpha}}\,, \quad E^{\mathbf{u}}\| \;=\; dx^\mu v_\mu{}^{\mathbf{u}}\,, \tag{4.11}$$

while the antisymmetric tensor can be identified in a superspace 2–form B:

$$B\| \;=\; \frac{1}{2}\,dx^\mu dx^\nu b_{\nu\mu}. \tag{4.12}$$

The remaining component fields, a real scalar and 4 helicity 1/2 fields, are identified in the supersymmetry transforms of the vielbein and 2–form, that is in torsion ($T^{\mathcal{A}} = DE^{\mathcal{A}}$) and 3–form ($H = dB$) components. The Bianchi identities satisfied by these objects are

$$DT^{\mathcal{A}} \;=\; E^{\mathcal{B}}R_{\mathcal{B}}{}^{\mathcal{A}}\,, \qquad\qquad dH \;=\; 0\,, \tag{4.13}$$

and, displaying the 4–form coefficients,

$$\left({}_{\mathcal{DCBA}}\right)_H \quad : \quad E^{\mathcal{A}}E^{\mathcal{B}}E^{\mathcal{C}}E^{\mathcal{D}}\left(2\mathcal{D}_{\mathcal{D}}H_{\mathcal{CBA}} + 3T_{\mathcal{DC}}{}^{\mathcal{F}}H_{\mathcal{FBA}}\right) = 0. \tag{4.14}$$

By putting constraints on torsion and 3–form we have to solve two problems at the same time: first, we have to reduce the huge number of superfluous independent fields contained in these geometrical objects, and second, we have to make sure that the antisymmetric tensor takes part of the *same* multiplet as $e_\mu{}^a$, $\psi_{\mu}{}^{\alpha}_{\mathrm{A}}$, $\bar{\psi}_{\mu}{}^{\mathrm{A}}_{\dot{\alpha}}$, $v_\mu{}^{\mathbf{u}}$ (soldering mechanism).

Indeed, the biggest problem in finding a geometrical description of an off-shell supersymmetric theory is to find suitable covariant constraints which do reduce this number but do not imply equations of motion for the remaining fields. There are several approaches to this question. One of them is based on conventional constraints, which resume to suitable redefinitions of the vielbein and connection and which do not imply equations of motion [76]. However, such redefinitions leave intact torsion components with 0 canonical dimension and there is no general recipes to indicate how these torsion components have to be constrained. A simpler manner of constraining 0 dimensional torsion components together with conventional constraints give rise to the so-called *natural constraints*, which were analyzed in a systematic way both in ordinary extended superspace [77] and in central charge superspace [73].

The geometrical description of the N-T multiplet is based on a set of natural constraints in central charge superspace with structure group $SL(2,\mathbb{C}) \otimes U(4)$. The generalizations of the canonical dimension 0 "trivial constraints" [77] to central charge superspace are

$$T^{\mathrm{CB}a}_{\gamma\beta} \;=\; 0\,, \qquad T^{\mathrm{C}\dot{\beta}a}_{\gamma\mathrm{B}} \;=\; -2i\delta^{\mathrm{C}}_{\mathrm{B}}(\sigma^a\epsilon)_\gamma{}^{\dot{\beta}}\,, \qquad T^{\dot{\gamma}\dot{\beta}a}_{\mathrm{CB}} \;=\; 0\,, \tag{4.15}$$

$$T^{\mathrm{CB}\mathbf{u}}_{\gamma\beta} \;=\; \epsilon_{\gamma\beta}T^{[\mathrm{CB}]\mathbf{u}}\,, \qquad T^{\mathrm{C}\dot{\beta}\mathbf{u}}_{\gamma\;\mathrm{B}} \;=\; 0\,, \qquad T^{\dot{\gamma}\dot{\beta}\mathbf{u}}_{\mathrm{CB}} \;=\; \epsilon^{\dot{\gamma}\dot{\beta}}T_{[\mathrm{CB}]}{}^{\mathbf{u}}\,. \tag{4.16}$$

As explained in detail in the article [13], the soldering is achieved by requiring some analogous, "mirror"-constraints for the 2–form sector. Besides the -1/2 dimensional constraints $H_{\gamma\,\beta\alpha}^{\mathrm{CBA}} = H_{\gamma\,\beta\mathrm{A}}^{\mathrm{CB}\dot\alpha} = H_{\gamma\,\mathrm{BA}}^{\mathrm{C}\dot\beta\dot\alpha} = H_{\mathrm{CBA}}^{\dot\gamma\dot\beta\dot\alpha} = 0$, we impose

$$H_{\gamma\beta a}^{\mathrm{CB}} \;=\; 0\,, \qquad H_{\gamma\mathrm{B}a}^{\mathrm{C}\dot\beta} \;=\; -2i\delta_{\mathrm{B}}^{\mathrm{C}}(\sigma_a\epsilon)_\gamma{}^{\dot\beta}L\,, \qquad H_{\mathrm{CB}a}^{\dot\gamma\dot\beta} \;=\; 0\,, \tag{4.17}$$

$$H_{\beta\alpha\mathbf{u}}^{\mathrm{BA}} = \epsilon_{\beta\alpha}H_{\mathbf{u}}{}^{[\mathrm{BA}]}\,, \qquad H_{\gamma\,\mathrm{B}\mathbf{u}}^{\mathrm{C}\dot\beta} = 0\,, \qquad H_{\mathrm{BA}\mathbf{u}}^{\dot\beta\dot\alpha} = \epsilon^{\dot\beta\dot\alpha}H_{\mathbf{u}[\mathrm{BA}]}\,, \tag{4.18}$$

with L a real superfield. The physical scalar ϕ of the multiplet, called also graviscalar, is identified in this superfield, parameterized as $L = e^{2\phi}$. In turn, the helicity 1/2 fields, called also gravigini fields, are identified as usual [78], [79], [68] in the 1/2–dimensional torsion component

$$\epsilon^{\beta\gamma}T_{\gamma\,\beta\dot\alpha}^{\mathrm{CBA}} \;=\; 2T^{[\mathrm{CBA}]}{}_{\dot\alpha}, \qquad \epsilon_{\dot\beta\dot\gamma}T_{\mathrm{C\,BA}}^{\dot\gamma\dot\beta\alpha} \;=\; 2T_{[\mathrm{CBA}]}{}^{\alpha}. \tag{4.19}$$

The scalar, the four helicity 1/2 fields, together with the gauge–fields defined in (4.11) and (4.12) constitute the N-T on-shell $N = 4$ supergravity multiplet. However, the 0 dimensional natural constraints listed above are not sufficient to insure that these are the *only* fields transforming into each-other by supergravity transformations. The elimination of a big number of superfluous fields is achieved by assuming the constraints

$$\mathcal{D}^{\mathrm{D}\alpha}T_{[\mathrm{CBA}]\alpha} \;=\; 0\,, \qquad \mathcal{D}_{\mathrm{D}\dot\alpha}T^{[\mathrm{CBA}]\dot\alpha} \;=\; 0\,, \tag{4.20}$$

and

$$T_{\mathbf{z}\mathcal{B}}{}^{\mathcal{A}} = 0, \tag{4.21}$$

as well as all possible compatible conventional constraints[3] [77], [73].

It is worthwhile to note that even at this stage the assumptions are not sufficient to constrain the geometry to the N-T multiplet. This setup allows to give a geometrical description at least of the coupling of $N = 4$ supergravity with antisymmetric tensor to six copies of $N = 4$ Yang-Mills multiplets [11]. Nevertheless, they are strong enough to put the underlying multiplet on-shell. In order to see this, one can easily verify that the dimension 1 Bianchi identities $\left({}^{\dot\delta\,\dot\gamma\,\mathrm{B}\alpha}_{\mathrm{DC}\beta\mathrm{A}}\right)_T$ and $\left({}^{\dot\delta\,\dot\gamma\,\dot\beta\,\mathrm{A}}_{\mathrm{DCB}\dot\alpha}\right)_T$ for the torsion as well as their complex conjugates imply

$$\begin{aligned}
\mathcal{D}_\delta^{\mathrm{D}}T_{[\mathrm{CBA}]\alpha} \;&=\; -i\delta_{\mathrm{CBA}}^{\mathrm{DEF}}G_{(\delta\alpha)[\mathrm{EF}]}\,, \qquad & \mathcal{D}_{\mathrm{D}}^{\dot\delta}T^{[\mathrm{CBA}]\dot\alpha} \;&=\; -i\delta_{\mathrm{DEF}}^{\mathrm{CBA}}G^{(\dot\delta\dot\alpha)[\mathrm{EF}]}\,, \\
\mathcal{D}_{\mathrm{D}}^{\dot\delta}T_{[\mathrm{CBA}]\alpha} \;&=\; P_\alpha{}^{\dot\delta}{}_{[\mathrm{DCBA}]}\,, \qquad & \mathcal{D}_\delta^{\mathrm{D}}T^{[\mathrm{CBA}]\dot\alpha} \;&=\; P_\delta{}^{\dot\alpha[\mathrm{DCBA}]}\,,
\end{aligned} \tag{4.22}$$

with G and P a priori some arbitrary superfields. Let us write one of the last relations as

$$\sum_{\mathrm{DC}}\mathcal{D}_{\mathrm{D}}^{\dot\delta}T_{[\mathrm{CBA}]\alpha} \;=\; 0, \tag{4.23}$$

take its spinorial derivative $\mathcal{D}_\varepsilon^{\mathrm{E}}$

$$\sum_{\mathrm{DC}}\left(\left\{\mathcal{D}_\varepsilon^{\mathrm{E}},\mathcal{D}_{\mathrm{D}}^{\dot\delta}\right\}T_{[\mathrm{CBA}]\alpha} - \mathcal{D}_{\mathrm{D}}^{\dot\delta}\left(\mathcal{D}_\varepsilon^{\mathrm{E}}T_{[\mathrm{CBA}]\alpha}\right)\right) \;=\; 0, \tag{4.24}$$

and observe that the antisymmetric part of this relation in the indices ε and α gives rise to Dirac equation for the helicity 1/2 fields, that is $\partial^{\alpha\dot\delta}T_{[\mathrm{CBA}]\alpha} = 0$ in the linear approach.

It turns out, that there is a simple solution of both the Bianchi identities of the torsion and 3–form, which satisfies the above mentioned constraints and reproduce the N-T multiplet. The non-zero torsion and 3–form components for this solution are listed in the appendix, we will

[3] see equations (F.1) in the appendix

concentrate here on its properties which are essential for the identification of the multiplet and the derivation of the equations of motion for the component fields.

Recall that the particularity of this solution is based on the identification of the scalar superfield ϕ in the 0 dimensional torsion and 3–form components containing a central charge index

$$T^{[\mathrm{BA}]\mathbf{u}} \;=\; 4e^{\phi}\mathfrak{t}^{[\mathrm{BA}]\mathbf{u}}\,, \qquad T_{[\mathrm{BA}]}{}^{\mathbf{u}} \;=\; 4e^{\phi}\mathfrak{t}_{[\mathrm{BA}]}{}^{\mathbf{u}}\,, \tag{4.25}$$

$$H_{\mathbf{u}}{}^{[\mathrm{BA}]} \;=\; 4e^{\phi}\mathfrak{h}_{\mathbf{u}}{}^{[\mathrm{BA}]}\,, \qquad H_{\mathbf{u}[\mathrm{BA}]} \;=\; 4e^{\phi}\mathfrak{h}_{\mathbf{u}[\mathrm{BA}]}\,, \tag{4.26}$$

with $\mathfrak{t}^{[\mathrm{CB}]\mathbf{u}}$, $\mathfrak{t}_{[\mathrm{CB}]}{}^{\mathbf{u}}$, $\mathfrak{h}_{\mathbf{u}}{}^{[\mathrm{BA}]}$, $\mathfrak{h}_{\mathbf{u}[\mathrm{BA}]}$ constant matrix elements satisfying the self–duality relations

$$\mathfrak{t}^{[\mathrm{DC}]\mathbf{u}} \;=\; \frac{q}{4}\,\varepsilon^{\mathrm{DCBA}}\mathfrak{t}_{[\mathrm{BA}]}{}^{\mathbf{u}}\,, \qquad \mathfrak{h}_{\mathbf{u}}{}^{[\mathrm{BA}]} \;=\; \frac{q}{2}\,\mathfrak{h}_{\mathbf{u}[\mathrm{DC}]}\varepsilon^{\mathrm{DCBA}} \qquad \text{with} \quad q=\pm 1. \tag{4.27}$$

Note, that these relations look similar to *some* of the properties of the 6 real, antisymmetric 4×4 matrices α^n, β^n, $(n=1,2,3)$ of $SU(2)\otimes SU(2)$ [66], [80], which appear in the component formulation of $N=4$ supergravity theories. Indeed, if we define the matrices

$$\mathfrak{t} \doteq \begin{pmatrix} \mathfrak{t}^{[\mathrm{DC}]\mathbf{u}} \\ \mathfrak{t}_{[\mathrm{DC}]}{}^{\mathbf{u}} \end{pmatrix}, \qquad \mathfrak{h} \doteq \begin{pmatrix} \mathfrak{h}_{\mathbf{u}[\mathrm{DC}]} & \mathfrak{h}_{\mathbf{u}}{}^{[\mathrm{DC}]} \end{pmatrix} \qquad \text{and} \tag{4.28}$$

$$\Sigma \doteq \begin{pmatrix} 0 & \frac{q}{2}\varepsilon^{\mathrm{DCBA}} \\ \frac{q}{2}\varepsilon_{\mathrm{DCBA}} & 0 \end{pmatrix}, \qquad \mathbf{1} \doteq \begin{pmatrix} \frac{1}{0}\delta^{\mathrm{DC}}_{\mathrm{BA}} & 0 \\ 0 & \frac{1}{2}\delta^{\mathrm{BA}}_{\mathrm{DC}} \end{pmatrix} \qquad \text{satisfying} \qquad \Sigma^2=\mathbf{1}\,, \tag{4.29}$$

then the properties of the matrix elements $\mathfrak{t}^{[\mathrm{CB}]\mathbf{u}}$, $\mathfrak{t}_{[\mathrm{CB}]}{}^{\mathbf{u}}$, $\mathfrak{h}_{\mathbf{u}}{}^{[\mathrm{BA}]}$, $\mathfrak{h}_{\mathbf{u}[\mathrm{BA}]}$ can be resumed in a compact way as follows:

$$\Sigma\mathfrak{t} \;=\; \mathfrak{t}\,, \qquad \mathfrak{h}\Sigma \;=\; \mathfrak{h}\,, \tag{4.30}$$

$$\mathfrak{t}\mathfrak{h} \;=\; \mathbf{1}+\Sigma\,, \qquad (\mathfrak{h}\mathfrak{t})_{\mathbf{u}}{}^{\mathbf{v}} \;=\; 2\delta^{\mathbf{v}}_{\mathbf{u}}\,. \tag{4.31}$$

Recall, however that we didn't fix a priori the number of the central charge coordinates in the superspace. The interesting feature of the above properties is that taking the trace of relations in (4.31) one finds $\delta^{\mathbf{u}}_{\mathbf{u}}=6$, that is the number of central charge indices - and thus, the number of the vector gauge–fields $v_m{}^{\mathbf{u}}$ - is determined to be 6.

These matrices serve as converters between the central charge basis (indices $\mathbf{u}$) and the $SU(4)$ basis in the antisymmetric representation (indices $[\mathrm{DC}]$). In particular, for the 6 vector gauge fields $v_m{}^{\mathbf{u}}$ of the N-T multiplet there is an alternative basis, called the $SU(4)$ basis, defined by

$$\begin{pmatrix} V_{\mu[\mathrm{DC}]} & V_{\mu}{}^{[\mathrm{DC}]} \end{pmatrix} \;\doteq\; v_{\mu}{}^{\mathbf{u}}\begin{pmatrix} \mathfrak{h}_{\mathbf{u}[\mathrm{DC}]} & \mathfrak{h}_{\mathbf{u}}{}^{[\mathrm{DC}]} \end{pmatrix}, \tag{4.32}$$

where the two components are connected by the self–duality relations

$$V_{\mu}{}^{[\mathrm{DC}]} \;=\; \frac{q}{2}\,\varepsilon^{\mathrm{DCBA}}V_{\mu[\mathrm{BA}]}. \tag{4.33}$$

Moreover, if we look at self–duality properties (4.27) as the lifting and lowering of $SU(4)$ indices with metric $\frac{q}{2}\varepsilon_{\mathrm{DCBA}}$, then a corresponding metric in the central charge basis can be defined by

$$\mathfrak{g}_{\mathbf{vu}} \;=\; \frac{q}{2}\,\varepsilon_{\mathrm{DCBA}}\,\mathfrak{h}_{\mathbf{v}}{}^{[\mathrm{DC}]}\,\mathfrak{h}_{\mathbf{u}}{}^{[\mathrm{BA}]}, \qquad \mathfrak{g}^{\mathbf{vu}} \;=\; \frac{q}{2}\,\varepsilon_{\mathrm{DCBA}}\,\mathfrak{t}^{[\mathrm{DC}]\mathbf{v}}\,\mathfrak{t}^{[\mathrm{BA}]\mathbf{u}}, \tag{4.34}$$

satisfying

$$\mathfrak{g}_{\mathbf{uw}}\mathfrak{g}^{\mathbf{wv}} \;=\; \delta^{\mathbf{v}}_{\mathbf{u}}\,. \tag{4.35}$$

These are the objects which are found to connect torsion and 3–form components containing at least one central charge index

$$H_{\mathcal{DC}\mathbf{u}} \;=\; T_{\mathcal{DC}}{}^{\mathbf{z}}\mathfrak{g}_{\mathbf{zu}}, \qquad T_{\mathcal{DC}}{}^{\mathbf{u}} \;=\; H_{\mathcal{DC}\mathbf{z}}\mathfrak{g}^{\mathbf{zu}}, \tag{4.36}$$

insuring the soldering of the two geometries.

The four helicity 1/2 fields $T_{[\mathrm{CBA}]\alpha}$, $T^{[\mathrm{CBA}]\dot{\alpha}}$ turn out to be equivalent to the fermionic partner of the graviscalar ϕ

$$\lambda_\alpha^{\mathrm{A}} = 2\mathcal{D}_\alpha^{\mathrm{A}}\phi\,, \qquad \bar{\lambda}_{\mathrm{A}}^{\dot{\alpha}} = 2\mathcal{D}_{\mathrm{A}}^{\dot{\alpha}}\phi\,, \tag{4.37}$$

since the following duality relation holds in this $N = 4$ case:

$$T_{[\mathrm{CBA}]\alpha} = q\varepsilon_{\mathrm{CBAF}}\bar{\lambda}_\alpha^{\mathrm{F}} \qquad T^{[\mathrm{CBA}]\dot{\alpha}} = q\varepsilon^{\mathrm{CBAF}}\lambda_{\mathrm{F}}^{\dot{\alpha}}\,. \tag{4.38}$$

It is the soldering mechanism between the geometry of supergravity and the geometry of the 2–form, that determines how the superfields G and P in the spinorial derivatives of this helicity 1/2 fields (4.22) are related to the component fields of the multiplet. In particular, we find that the superfields G are related to the covariant field strength of the graviphotons $F_{ba}{}^{\mathbf{u}}$

$$G_{(\beta\alpha)[\mathrm{BA}]} = -2ie^{-\phi}F_{(\beta\alpha)}{}^{\mathbf{u}}\mathfrak{h}_{\mathbf{u}[\mathrm{BA}]}\,, \qquad G^{(\dot{\beta}\dot{\alpha})[\mathrm{BA}]} = -2ie^{-\phi}F^{(\dot{\beta}\dot{\alpha})\mathbf{u}}\mathfrak{h}_{\mathbf{u}}{}^{[\mathrm{BA}]}\,, \tag{4.39}$$

whereas the superfields P contain the dual field strength of the antisymmetric tensor and the derivative of the scalar:

$$\mathcal{D}_\delta^{\mathrm{D}}T^{[\mathrm{CBA}]\dot{\alpha}} = q\varepsilon^{\mathrm{DCBA}}P_\delta{}^{\dot{\alpha}}\,, \quad \text{with} \quad P_a = 2i\mathcal{D}_a\phi + e^{-2\phi}H_a^* - \frac{3}{4}\lambda^{\mathrm{A}}\sigma_a\bar{\lambda}_{\mathrm{A}}\,, \tag{4.40}$$

$$\mathcal{D}_{\mathrm{D}}^{\dot{\alpha}}T_{[\mathrm{CBA}]\delta} = q\varepsilon_{\mathrm{DCBA}}\bar{P}_\delta{}^{\dot{\alpha}}\,, \quad \text{with} \quad \bar{P}_a = 2i\mathcal{D}_a\phi - e^{-2\phi}H_a^* + \frac{3}{4}\lambda^{\mathrm{A}}\sigma_a\bar{\lambda}_{\mathrm{A}}\,, \tag{4.41}$$

where we can note that the relations

$$P_a + \bar{P}_a = 4i\mathcal{D}_a\phi\,, \qquad P_a - \bar{P}_a = 2e^{-2\phi}H_a^* - \frac{3}{2}\lambda^{\mathrm{A}}\sigma_a\bar{\lambda}_{\mathrm{A}} \tag{4.42}$$

allow to separate the dual field strength of the antisymmetric tensor and the derivative of the scalar (as "real" and "imaginary" part of P).

Finally, let us precise that the representation of the structure group in the central charge sector is trivial, $\Phi_{\mathbf{u}}{}^{\mathbf{z}} = 0$, while the $U(4)$ part $\Phi^{\mathrm{B}}{}_{\mathrm{A}}$ of the $SL(2,\mathbb{C}) \otimes U(4)$ connection

$$\Phi_{\beta\mathrm{A}}^{\mathrm{B}\alpha} = \delta_{\mathrm{A}}^{\mathrm{B}}\Phi_\beta{}^\alpha + \delta_\beta^\alpha\Phi^{\mathrm{B}}{}_{\mathrm{A}} \qquad \Phi_{\mathrm{B}\dot{\alpha}}^{\dot{\beta}\mathrm{A}} = \delta_{\mathrm{B}}^{\mathrm{A}}\Phi^{\dot{\beta}}{}_{\dot{\alpha}} - \delta_{\dot{\alpha}}^{\dot{\beta}}\Phi^{\mathrm{A}}{}_{\mathrm{B}} \tag{4.43}$$

is determined to be

$$\Phi^{\mathrm{B}}{}_{\mathrm{A}} = a^{\mathrm{B}}{}_{\mathrm{A}} + \chi^{\mathrm{B}}{}_{\mathrm{A}}\,, \tag{4.44}$$

with $a^{\mathrm{B}}{}_{\mathrm{A}}$ pure gauge and $\chi^{\mathrm{B}}{}_{\mathrm{A}}$ a supercovariant 1–form on the superspace with components

$$\chi_c{}^{\mathrm{B}}{}_{\mathrm{A}} = \frac{1}{4}\delta_{\mathrm{A}}^{\mathrm{B}}\left(ie^{-2\phi}H_c^* - \frac{i}{4}\lambda^{\mathrm{F}}\sigma_c\bar{\lambda}_{\mathrm{F}}\right) - \frac{i}{8}(\lambda^{\mathrm{B}}\sigma_c\bar{\lambda}_{\mathrm{A}})\,,$$

$$\chi_\gamma^{\mathrm{CB}}{}_{\mathrm{A}} = \frac{1}{4}\delta_{\mathrm{A}}^{\mathrm{B}}\lambda_\gamma^{\mathrm{C}}\,, \qquad \chi_{\mathrm{C}}^{\dot{\gamma}\mathrm{B}}{}_{\mathrm{A}} = -\frac{1}{4}\delta_{\mathrm{A}}^{\mathrm{B}}\bar{\lambda}_{\mathrm{C}}^{\dot{\gamma}}\,, \qquad \chi_{\mathbf{u}}{}^{\mathrm{B}}{}_{\mathrm{A}} = 0\,. \tag{4.45}$$

This situation is analogous to the case of the 16+16 $N = 1$ supergravity multiplet which is obtained from the reducible 20+20 multiplet, described on superspace with structure group $SL(2,\mathbb{C}) \otimes U(1)$, by "breaking" the $U(1)$ symmetry [81]. By eliminating this $U(4)$ part from the $SL(2,\mathbb{C}) \otimes U(4)$ connection and putting the pure gauge part a to zero, one can define covariant derivatives for $SL(2,\mathbb{C})$

$$\hat{D}u^{\mathcal{A}} = Du^{\mathcal{A}} - \chi_{\mathcal{B}}{}^{\mathcal{A}}u^{\mathcal{B}} \qquad \hat{D}u_{\mathcal{A}} = Du_{\mathcal{A}} + \chi_{\mathcal{A}}{}^{\mathcal{B}}u_{\mathcal{B}} \tag{4.46}$$

used in the articles [13] and [12]. Here of course $\chi_{\mathcal{B}}{}^{\mathcal{A}}$ is defined in such a way that its only non-zero components are $\chi_{\beta\mathrm{A}}^{\mathrm{B}\alpha} = \delta_\beta^\alpha\chi^{\mathrm{B}}{}_{\mathrm{A}}$ and $\chi_{\mathrm{B}\dot{\alpha}}^{\dot{\beta}\mathrm{A}} = -\delta_{\dot{\alpha}}^{\dot{\beta}}\chi^{\mathrm{A}}{}_{\mathrm{B}}$. Recall that this redefinition of the connection affects torsion and curvature components in the following way:

$$\hat{T}_{\mathcal{CB}}{}^{\mathcal{A}} = T_{\mathcal{CB}}{}^{\mathcal{A}} - \chi_{\mathcal{CB}}{}^{\mathcal{A}} + (-)^{cb}\chi_{\mathcal{BC}}{}^{\mathcal{A}}, \tag{4.47}$$

$$\hat{R}_{\mathcal{DC}}{}^{\mathrm{B}}{}_{\mathrm{A}} = 0\,. \tag{4.48}$$

In the next section we derive the equations of motion for all the component fields of the N-T multiplet using its geometrical description presented above.

4.4 Equations of motion in terms of supercovariant quantities

The problem of the derivation of field equations of motion without the knowledge of a Lagrangian, using considerations on representations of the symmetry group, was considered a long time ago [82], [83]. The question is particularly interesting for supersymmetric theories and there are various approaches which have been developed. Let us mention for example the procedure based on projection operators selecting irreducible representations out of superfield with arbitrary external spin [84]. About the same period Wess and Zumino suggested the use of differential geometry in superspace to reach better understanding of supersymmetric Yang-Mills and supergravity theories. The techniques used in this approach allowed to work out a new method for deriving equations of motion, namely looking at consequences of covariant constraints, which correspond to on-shell field content of a representation of the supersymmetry algebra.

In order to illustrate the method let us recall as briefly as possible the simplest example, the $N = 1$ Yang-Mills theory described on superspace considering the geometry of a Lie algebra valued 1–form $\mathcal{A}$ [85], [72]. Under a gauge transformation, parameterized by g, the gauge potential transforms as $\mathcal{A} \mapsto g^{-1}\mathcal{A}g - g^{-1}dg$ and its field strength $\mathcal{F} = d\mathcal{A} + \mathcal{A}\mathcal{A}$ satisfies the Bianchi identity $D\mathcal{F} = 0$. In order to describe the on-shell multiplet one constrains the geometry by putting $\mathcal{F}_{\alpha\beta} = \mathcal{F}_\alpha{}^{\dot\beta} = \mathcal{F}^{\dot\alpha\dot\beta} = 0$. Then the Bianchi identities are satisfied if and only if all the components of the field strength $\mathcal{F}$ can be expressed in terms of two spinor superfields $\mathcal{W}_\alpha$, $\bar{\mathcal{W}}^{\dot\alpha}$ and their spinor derivatives:

$$\mathcal{F}_{\beta a} = i(\sigma_a\bar{\mathcal{W}})_\beta\,, \qquad \mathcal{F}^{\dot\beta}{}_a = -i(\bar\sigma_a\mathcal{W})^{\dot\beta}\,, \tag{4.49}$$

$$\mathcal{F}_{(\beta\alpha)} = -\frac{1}{2}\mathcal{D}_{(\beta}\mathcal{W}_{\alpha)}\,, \qquad \mathcal{F}^{(\dot\beta\dot\alpha)} = \frac{1}{2}\mathcal{D}^{(\dot\beta}\bar{\mathcal{W}}^{\dot\alpha)}\,, \tag{4.50}$$

and the gaugino superfields $\mathcal{W}_\alpha$, $\bar{\mathcal{W}}^{\dot\alpha}$ satisfy

$$\mathcal{D}_\alpha\bar{\mathcal{W}}^{\dot\alpha} \;=\; 0\,, \qquad \mathcal{D}^{\dot\alpha}\mathcal{W}_\alpha \;=\; 0\,, \tag{4.51}$$

$$\mathcal{D}^\alpha\mathcal{W}_\alpha \;=\; \mathcal{D}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}\,. \tag{4.52}$$

The components of the multiplet are thus identified as follows: the vector gauge field in the super 1–form $\mathcal{A}|\!| = i\mathrm{d}x^m a_m$, the gaugino component field as lowest component of the gaugino superfield $\mathcal{W}_\alpha| = -i\lambda_\alpha$, $\bar{\mathcal{W}}^{\dot\alpha}| = i\lambda^{\dot\alpha}$, and the auxiliary field in their derivatives $\mathcal{D}^\alpha\mathcal{W}_\alpha| = \mathcal{D}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}| = -2D$.

Note that the supplementary constraint

$$\mathcal{D}^\alpha\mathcal{W}_\alpha \;=\; \mathcal{D}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha} \;=\; 0 \tag{4.53}$$

puts this multiplet on-shell. It is a superfield equation and contains all the component field equations of motion. First of all it eliminates the auxiliary field D and we can derive the equations of motion for the remaining fields by successively differentiating it. We obtain the Dirac equation for the gaugino

$$\mathcal{D}^{\dot\alpha}\left(\mathcal{D}^\alpha\mathcal{W}_\alpha\right) \;=\; -2i\mathcal{D}^{\alpha\dot\alpha}\mathcal{W}_\alpha \;=\; 0\,, \tag{4.54}$$

$$\mathcal{D}_\alpha\left(\mathcal{D}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}\right) \;=\; -2i\mathcal{D}_{\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha} \;=\; 0\,, \tag{4.55}$$

and from this we derive the relations

$$\mathcal{D}_\beta\left(\mathcal{D}^{\alpha\dot\alpha}\mathcal{W}_\alpha\right) \;=\; -2\mathcal{D}^{\alpha\dot\alpha}\mathcal{F}_{(\beta\alpha)} + 2i\{\mathcal{W}_\beta, \bar{\mathcal{W}}^{\dot\alpha}\} \;=\; 0\,, \tag{4.56}$$

$$\mathcal{D}^{\dot\beta}\left(\mathcal{D}_{\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}\right) \;=\; 2\mathcal{D}_{\alpha\dot\alpha}\mathcal{F}^{(\dot\beta\dot\alpha)} - 2i\{\bar{\mathcal{W}}^{\dot\beta}, \mathcal{W}_\alpha\} \;=\; 0\,, \tag{4.57}$$

which correspond to the well-known Bianchi identities $\mathcal{D}_{\alpha\dot\beta}\mathcal{F}^{(\dot\beta\dot\alpha)} - \mathcal{D}^{\beta\dot\alpha}\mathcal{F}_{(\beta\alpha)} = 0$ and equations of motion $\mathcal{D}_{\alpha\dot\beta}\mathcal{F}^{(\dot\beta\dot\alpha)} + \mathcal{D}^{\beta\dot\alpha}\mathcal{F}_{(\beta\alpha)} = 2i\{\mathcal{W}_\alpha, \bar{\mathcal{W}}^{\dot\beta}\}$ for the vector gauge field.

The case of supergravity is similar to this, the gravigino superfields $T_{[\text{CBA}]\alpha}$, $T^{[\text{CBA}]\dot\alpha}$ (or $\lambda_\alpha^{\text{A}}$, $\bar\lambda_{\text{A}}^{\dot\alpha}$ in (4.38)) play an analogous rôle to the gaugino superfields $\mathcal{W}_\alpha$, $\bar{\mathcal{W}}^{\dot\alpha}$. In order to derive the free equations of motion of component fields in a supergravity theory it is sufficient to consider only the linearized version [86], [87], [79] and the calculations are simple. Considering the full theory one obtains all the nonlinear terms which arise in equations of motion derived from a Lagrangian in component formalism.

Recall that the dimension 1 Bianchi identities in the supergravity sector imply the relations (4.22) for the spinor derivatives of the gravigino superfields. These properties can be written equivalently as

$$\begin{aligned}
\textstyle\sum_{\text{DC}} \mathcal{D}_{\text{D}}^{\dot\delta} T_{[\text{CBA}]\alpha} &= 0\,, & \textstyle\sum^{\text{DC}} \mathcal{D}_\delta^{\text{D}} T^{[\text{CBA}]\dot\alpha} &= 0\,, \\
\mathcal{D}^{\text{D}\alpha} T_{[\text{CBA}]\alpha} &= 0\,, & \mathcal{D}_{\text{D}\dot\alpha} T^{[\text{CBA}]\dot\alpha} &= 0\,, \\
\mathcal{D}^{\text{D}}_{(\delta} T_{\alpha)[\text{CBA}]} - \tfrac{1}{4}\delta^{\text{DEF}}_{\text{CBA}} \mathcal{D}^{\text{G}}_{(\delta} T_{\alpha)[\text{GEF}]} &= 0\,, & \mathcal{D}^{(\dot\delta}_{\text{D}} T^{\dot\alpha)[\text{CBA}]} - \tfrac{1}{4}\delta^{\text{CBA}}_{\text{DEF}} \mathcal{D}^{(\dot\delta}_{\text{G}} T^{\dot\alpha)[\text{GEF}]} &= 0\,,
\end{aligned} \tag{4.58}$$

and they are the $N = 4$ analogues of the relations (4.51) and (4.53) satisfied by the gaugino superfield corresponding to the on-shell Yang-Mills multiplet.

Therefore, by analogy to the Yang–Mills case, the equations of motion for the gravigini, the graviphoton, the scalar and the antisymmetric tensor can be deduced from the superfield relations (4.22) by taking successive covariant spinorial derivatives. Let us take the example of Dirac's equation, to clarify ideas. We start from (4.24). The anticommutator can be expressed using (4.4) and (4.5)

$$\{\mathcal{D}^{\text{E}}_\epsilon\,,\, \mathcal{D}^{\dot\delta}_{\text{D}}\} T_{[\text{CBA}]\alpha} = -T_{\epsilon\text{D}}^{\text{E}\dot\delta f}\mathcal{D}_f T_{[\text{CBA}]\alpha} - R_{\epsilon\text{D}\alpha}^{\text{E}\dot\delta\ \ \beta} T_{[\text{CBA}]\beta} \tag{4.59}$$

where we have used the results displayed in the appendix F for the torsion and curvature components, which make it possible to evaluate this expression explicitly. When we take the antisymmetric part in $(\alpha\,,\,\epsilon)$, the second term in (4.24) drops out because of the symmetry of $G_{(\delta\alpha)[\text{ED}]}$ in (4.22). Summing on D and E, we obtain (4.65) below.

Consider now all possible spinorial derivatives of relations (4.22). They are satisfied if and only if in addition to the dimension 1 results the following relations hold:

$$\mathcal{D}^{\text{C}}_\gamma G_{(\beta\alpha)[\text{BA}]} = \frac{1}{3}\delta^{\text{CF}}_{\text{BA}}\left[\frac{1}{3}\oint_{\gamma\beta\alpha} \mathcal{D}^{\text{E}}_\gamma G_{(\beta\alpha)[\text{EF}]} + \frac{i}{2}\sum_{\beta\alpha} \epsilon_{\gamma\beta}\lambda_{\dot\alpha\text{F}}\bar P_\alpha{}^{\dot\alpha}\right] \tag{4.60}$$

$$\mathcal{D}^{\dot\gamma}_{\text{C}} G^{(\dot\beta\dot\alpha)[\text{BA}]} = \frac{1}{3}\delta^{\text{BA}}_{\text{CF}}\left[\frac{1}{3}\oint^{\dot\gamma\dot\beta\dot\alpha} \mathcal{D}^{\dot\gamma}_{\text{E}} G^{(\dot\beta\dot\alpha)[\text{EF}]} + \frac{i}{2}\sum^{\dot\beta\dot\alpha} \epsilon^{\dot\gamma\dot\beta}\lambda^{\alpha\text{F}} P_\alpha{}^{\dot\alpha}\right] \tag{4.61}$$

$$\mathcal{D}^{\text{C}}_\delta G^{(\dot\beta\dot\alpha)[\text{BA}]} = \mathcal{D}_\delta{}^{(\dot\beta} T^{\dot\alpha)[\text{CBA}]} + U^{\text{C}}_{\delta}{}^{(\dot\alpha}_{\text{F}} T^{\dot\beta)[\text{FBA}]} - U^{\text{B}}_{\delta}{}^{(\dot\alpha}_{\text{F}} T^{\dot\beta)[\text{FAC}]} - U^{\text{A}}_{\delta}{}^{(\dot\alpha}_{\text{F}} T^{\dot\beta)[\text{FCB}]} \tag{4.62}$$

$$\mathcal{D}^{\dot\delta}_{\text{C}} G_{(\beta\alpha)[\text{BA}]} = \mathcal{D}_{(\beta}{}^{\dot\delta} T_{\alpha)[\text{CBA}]} - U^{\text{F}}_{(\alpha\text{C}}{}^{\dot\delta} T_{\beta)[\text{FBA}]} + U^{\text{F}}_{(\alpha\text{B}}{}^{\dot\delta} T_{\beta)[\text{FAC}]} + U^{\text{F}}_{(\alpha\text{A}}{}^{\dot\delta} T_{\beta)[\text{FCB}]} \tag{4.63}$$

and

$$\mathcal{D}_{\beta\dot\alpha} T^{[\text{CBA}]\dot\alpha} = \frac{3}{2} U^{\text{F}\dot\alpha}_{\beta\,\text{F}} T^{[\text{CBA}]}_{\dot\alpha} \tag{4.64}$$

$$\mathcal{D}^{\alpha\dot\beta} T_{[\text{CBA}]\alpha} = -\frac{3}{2} U^{\text{F}\dot\beta}_{\alpha\,\text{F}} T^{\alpha}_{[\text{CBA}]} \tag{4.65}$$

with $U^{\text{B}\dot\alpha}_{\beta\,\text{A}} = \frac{i}{4}(\lambda^{\text{B}}_\beta \bar\lambda^{\dot\alpha}_{\text{A}} - \frac{1}{2}\delta^{\text{B}}_{\text{A}}\lambda^{\text{F}}_\beta\bar\lambda^{\dot\alpha}_{\text{F}})$.

Equations of motion for the helicity 1/2 fields.

Note first, that all these relations are implied also by Bianchi identities at dim 3/2. Secondly, note that the last equations, (4.64) and (4.65), are the Dirac equations for the spin 1/2 fields, which may be written in terms of the fields λ in the following way:

$$\mathcal{D}_{\beta\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_{\mathrm{A}} = ie^{-2\phi}H^*_{\beta\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_{\mathrm{A}} + \frac{9i}{8}(\bar{\lambda}_{\mathrm{A}}\bar{\lambda}_{\mathrm{F}})\lambda^{\mathrm{F}}_{\beta}, \tag{4.66}$$

$$\mathcal{D}^{\alpha\dot{\beta}}\lambda^{\mathrm{A}}_{\alpha} = -ie^{-2\phi}H^{*\alpha\dot{\beta}}\lambda^{\mathrm{A}}_{\alpha} + \frac{9i}{8}(\lambda^{\mathrm{A}}\lambda^{\mathrm{F}})\bar{\lambda}^{\dot{\beta}}_{\mathrm{F}}. \tag{4.67}$$

a. Consider the spinorial derivative $\mathcal{D}^{\mathrm{D}}_{\delta}$ of the Dirac equation (4.64). The derived identity is satisfied if and only if in addition to the results obtained till dimension 3/2 the following relations take place:

$$\mathcal{D}_{\alpha\dot{\alpha}}P^{\alpha\dot{\alpha}} - i\left(e^{-2\phi}H^*_{\alpha\dot{\alpha}} + \frac{1}{2}\lambda^{\mathrm{F}}_{\alpha}\bar{\lambda}_{\dot{\alpha}\mathrm{F}}\right)P^{\alpha\dot{\alpha}} + \frac{iq}{2}\varepsilon_{\mathrm{DCBA}}G^{(\dot{\beta}\dot{\alpha})[\mathrm{DC}]}G_{(\dot{\beta}\dot{\alpha})}{}^{[\mathrm{BA}]} = 0 \tag{4.68}$$

$$\sum_{\beta\alpha}\left[\mathcal{D}_{\beta\dot{\alpha}}P_{\alpha}{}^{\dot{\alpha}} - i\left(e^{-2\phi}H^*_{\beta\dot{\alpha}} + \lambda^{\mathrm{F}}_{\beta}\bar{\lambda}_{\dot{\alpha}\mathrm{F}}\right)P_{\alpha}{}^{\dot{\alpha}}\right] = 0. \tag{4.69}$$

b. Consider the spinorial derivative $\mathcal{D}^{\dot{\delta}}_{\mathrm{D}}$ of (4.65). The identity is satisfied if and only if in addition to the results obtained till dimension 3/2 the following relations take place:

$$\mathcal{D}_{\alpha\dot{\alpha}}\bar{P}^{\alpha\dot{\alpha}} + i\left(e^{-2\phi}H^*_{\alpha\dot{\alpha}} + \frac{1}{2}\lambda^{\mathrm{F}}_{\alpha}\bar{\lambda}_{\dot{\alpha}\mathrm{F}}\right)\bar{P}^{\alpha\dot{\alpha}} + \frac{iq}{2}\varepsilon^{\mathrm{DCBA}}G_{(\beta\alpha)[\mathrm{DC}]}G^{(\beta\alpha)}{}_{[\mathrm{BA}]} = 0 \tag{4.70}$$

$$\sum^{\dot{\beta}\dot{\alpha}}\left[\mathcal{D}^{\alpha\dot{\beta}}\bar{P}_{\alpha}{}^{\dot{\alpha}} + i\left(e^{-2\phi}H^{*\alpha\dot{\beta}} + \lambda^{\mathrm{F}\alpha}\bar{\lambda}^{\dot{\beta}}_{\mathrm{F}}\right)\bar{P}_{\alpha}{}^{\dot{\alpha}}\right] = 0. \tag{4.71}$$

Equations of motion for the scalar.

Using properties (4.42) the equations of motion for the scalar can be deduced from the sum of the relations (4.68) and (4.70):

$$\begin{aligned} 2\mathcal{D}_a\left(\mathcal{D}^a\phi\right) &= e^{-4\phi}H^*_aH^{*a} - e^{-2\phi}H^*_a(\lambda^{\mathrm{A}}\sigma^a\bar{\lambda}_{\mathrm{A}}) \\ &\quad -\frac{3}{8}(\lambda^{\mathrm{B}}\lambda^{\mathrm{A}})(\bar{\lambda}_{\mathrm{B}}\bar{\lambda}_{\mathrm{A}}) - \frac{1}{2}e^{-2\phi}F_{ba[\mathrm{BA}]}F^{ba[\mathrm{BA}]}. \end{aligned} \tag{4.72}$$

This equation already shows that in the Lagrangian corresponding to these equations of motion the kinetic terms of the antisymmetric tensor and of the graviphotons are accompanied by exponentials in the scalar field.

By the way, the difference of relations (4.68) and (4.70) looks as

$$\mathcal{D}_aH^{*a} = \frac{1}{2}e^{2\phi}(\lambda^{\mathrm{A}}\sigma_a\bar{\lambda}_{\mathrm{A}})\mathcal{D}^a\phi + \frac{i}{2}F^{*ba[\mathrm{BA}]}F_{ba[\mathrm{BA}]}, \tag{4.73}$$

and it corresponds of course to the Bianchi identity satisfied by the antisymmetric tensor gauge field. The topological term $F^{*ba[\mathrm{BA}]}F_{ba[\mathrm{BA}]}$ is an indication of the intrinsic presence of Chern-Simons forms in the geometry. This feature is analogous to the case of the off-shell $N=2$ minimal supergravity multiplet containing an antisymmetric tensor [70]. It arises naturally in extended supergravity using the soldering mechanism with the geometry of a 2–form in central charge superspace.

Equations of motion for the antisymmetric tensor.

Note that relations (4.69) and (4.71) are the selfdual and respectively the anti-selfdual part of the equation of motion for the antisymmetric tensor. Putting these relations together, we obtain the equation of motion for the antisymmetric tensor:

$$\begin{aligned}\varepsilon_{dcba}\mathcal{D}^b H^{*a} &= \left[T_{dc}{}^{\alpha}_{\mathrm{A}}\lambda^{\mathrm{A}}_{\alpha} + T_{dc}{}^{\mathrm{A}}_{\dot\alpha}\bar\lambda^{\dot\alpha}_{\mathrm{A}}\right]e^{2\phi} - \frac{1}{2}H^*_{[d}(\lambda^{\mathrm{F}}\sigma_{c]}\bar\lambda_{\mathrm{F}}) \\ &\quad + \varepsilon_{dcba}\left[\frac{3}{4}\mathcal{D}^b(\lambda^{\mathrm{F}}\sigma^a\bar\lambda_{\mathrm{F}}) - (\mathcal{D}^b\phi)(\lambda^{\mathrm{F}}\sigma^a\bar\lambda_{\mathrm{F}}) + 4e^{-2\phi}(\mathcal{D}^b\phi)H^{*a}\right]e^{2\phi} \qquad (4.74)\end{aligned}$$

Consider the spinorial derivative $\mathcal{D}^{\dot\delta}_{\mathrm{D}}$ of the Dirac equation (4.64) and the spinorial derivative $\mathcal{D}^{\mathrm{D}}_{\delta}$ of (4.65). The identities obtained this way are satisfied if and only if in addition to the results obtained till dimension 3/2 the following relations hold:

$$\begin{aligned}4i\mathcal{D}_{\beta\dot\alpha}G^{(\dot\delta\dot\alpha)[\mathrm{BA}]} &= q\varepsilon^{\mathrm{BADC}}\left(G_{(\beta\alpha)[\mathrm{DC}]}P^{\alpha\dot\delta} + i\bar\lambda^{\dot\alpha}_{\mathrm{D}}\bar\lambda^{\dot\delta}_{\mathrm{C}}\bar P_{\beta\dot\alpha}\right) \\ &\quad - G^{(\dot\delta\dot\alpha)[\mathrm{BF}]}\lambda^{\mathrm{A}}_{\beta}\bar\lambda_{\mathrm{F}\dot\alpha} - G^{(\dot\delta\dot\alpha)[\mathrm{FA}]}\lambda^{\mathrm{B}}_{\beta}\bar\lambda_{\mathrm{F}\dot\alpha} \qquad (4.75)\end{aligned}$$

$$\begin{aligned}4i\mathcal{D}^{\alpha\dot\beta}G_{(\delta\alpha)[\mathrm{BA}]} &= q\varepsilon_{\mathrm{BADC}}\left(G^{(\dot\beta\dot\alpha)[\mathrm{DC}]}\bar P_{\delta\dot\alpha} + i\lambda^{\mathrm{D}}_{\alpha}\lambda^{\mathrm{C}}_{\delta}P^{\alpha\dot\beta}\right) \\ &\quad + G_{(\delta\alpha)[\mathrm{BF}]}\lambda^{\mathrm{F}\alpha}\bar\lambda^{\dot\beta}_{\mathrm{A}} + G_{(\delta\alpha)[\mathrm{FA}]}\lambda^{\mathrm{F}\alpha}\bar\lambda^{\dot\beta}_{\mathrm{B}}. \qquad (4.76)\end{aligned}$$

Equations of motion for the graviphotons.

Recall that the geometric soldering mechanism between supergravity and the geometry of the 3-form implies that the fields $G_{(\beta\alpha)[\mathrm{BA}]}$ and $G^{(\dot\beta\dot\alpha)[\mathrm{BA}]}$ are related to the covariant field strength of the graviphotons, $F^{\mathbf{u}}$, by (4.39). Then the previous lemma determines both the equations of motion and the Bianchi identities satisfied by the vector gauge fields of the multiplet:

$$\begin{aligned}\mathcal{D}_b F^{ba\mathbf{u}} &= -\frac{i}{2}\left[(P_b + \bar P_b)F^{ba\mathbf{u}} + (P_b - \bar P_b)F^{*ba\mathbf{u}}\right] \\ &\quad + \frac{i}{4}\left[P_b(\lambda^{\mathrm{B}}\sigma^{ba}\bar\lambda^{\mathrm{A}})\mathfrak{t}_{[\mathrm{BA}]}{}^{\mathbf{u}} + \bar P_b(\lambda_{\mathrm{B}}\bar\sigma^{ba}\bar\lambda_{\mathrm{A}})\mathfrak{t}^{[\mathrm{BA}]\mathbf{u}}\right]e^{\phi} \\ &\quad - \frac{i}{4}\left[(\lambda^{\mathrm{A}}\sigma^{dc}\sigma^a\bar\lambda_{\mathrm{B}})F_{dc}{}^{\mathbf{v}}\mathfrak{h}_{\mathbf{v}}{}^{[\mathrm{BF}]}\mathfrak{t}_{[\mathrm{FA}]}{}^{\mathbf{u}} + (\bar\lambda_{\mathrm{A}}\bar\sigma^{dc}\bar\sigma^a\lambda^{\mathrm{B}})F_{dc}{}^{\mathbf{v}}\mathfrak{h}_{\mathbf{v}[\mathrm{BF}]}\mathfrak{t}^{[\mathrm{FA}]\mathbf{u}}\right], \qquad (4.77) \\ \mathcal{D}_b F^{*ba\mathbf{u}} &= \frac{i}{4}\left[P_b(\lambda^{\mathrm{B}}\sigma^{ba}\bar\lambda^{\mathrm{A}})\mathfrak{t}_{[\mathrm{BA}]}{}^{\mathbf{u}} - \bar P_b(\lambda_{\mathrm{B}}\bar\sigma^{ba}\bar\lambda_{\mathrm{A}})\mathfrak{t}^{[\mathrm{BA}]\mathbf{u}}\right]e^{\phi} \\ &\quad - \frac{i}{4}\left[(\lambda^{\mathrm{A}}\sigma^{dc}\sigma^a\bar\lambda_{\mathrm{B}})F_{dc}{}^{\mathbf{v}}\mathfrak{h}_{\mathbf{v}}{}^{[\mathrm{BF}]}\mathfrak{t}_{[\mathrm{FA}]}{}^{\mathbf{u}} - (\bar\lambda_{\mathrm{A}}\bar\sigma^{dc}\bar\sigma^a\lambda^{\mathrm{B}})F_{dc}{}^{\mathbf{v}}\mathfrak{h}_{\mathbf{v}[\mathrm{BF}]}\mathfrak{t}^{[\mathrm{FA}]\mathbf{u}}\right]. \qquad (4.78)\end{aligned}$$

Further differentiating (4.75) and (4.76) one can obtain Bianchi identities for the gravitini and graviton, but here we would like to derive their equations of motion instead.

Equations of motion for the gravitini.

Unlike the equations of motion presented above, the equations of motion for the gravitini and the graviton are directly given by the superspace Bianchi identities, once the component fields are identified. For example, the Bianchi identities at dim 3/2 determinate the torsion components

$$T_{(\beta\alpha)}{}^{\alpha}_{\mathrm{A}} = \frac{1}{16}\bar P_{\beta\dot\alpha}\bar\lambda^{\dot\alpha}_{\mathrm{A}} \qquad T^{(\dot\delta\dot\gamma)}{}_{\beta\mathrm{D}} = \frac{1}{8}\bar P_{\beta}{}^{(\dot\delta}\bar\lambda^{\dot\gamma)}_{\mathrm{D}} + \frac{iq}{8}\varepsilon_{\mathrm{DCBA}}G^{(\dot\delta\dot\gamma)[\mathrm{CB}]}\lambda^{\mathrm{A}}_{\beta}, \qquad (4.79)$$

$$T^{(\dot\beta\dot\alpha)}{}^{\mathrm{A}}_{\dot\alpha} = \frac{1}{16}P^{\alpha\dot\beta}\lambda^{\mathrm{A}}_{\alpha} \qquad T_{(\delta\gamma)}{}^{\dot\beta\mathrm{D}} = \frac{1}{8}P_{(\delta}{}^{\dot\beta}\lambda^{\mathrm{D}}_{\gamma)} + \frac{iq}{8}\varepsilon^{\mathrm{DCBA}}G_{(\delta\gamma)[\mathrm{CB}]}\bar\lambda^{\dot\beta}_{\mathrm{A}}, \qquad (4.80)$$

and these components are sufficient to give the equations of motion for the gravitini:

$$\varepsilon^{dcba}(\bar{\sigma}_c T_{ba\mathrm{A}})^{\dot{\alpha}} = \frac{i}{4}(\bar{\lambda}_{\mathrm{A}}\bar{\sigma}^d\sigma^c\epsilon)^{\dot{\alpha}}\bar{P}_c + \frac{i}{2}(\bar{\sigma}^{ba}\bar{\sigma}^d\lambda^{\mathrm{F}})^{\dot{\alpha}}F_{ba[\mathrm{AF}]}e^{-\phi}\,, \tag{4.81}$$

$$\varepsilon^{dcba}(\sigma_c T_{ba}{}^{\mathrm{A}})_\alpha = -\frac{i}{4}(\lambda^{\mathrm{A}}\sigma^d\bar{\sigma}^c\epsilon)_\alpha P_c - \frac{i}{2}(\sigma^{ba}\sigma^d\bar{\lambda}_{\mathrm{F}})_\alpha F_{ba}{}^{[\mathrm{AF}]}e^{-\phi}\,. \tag{4.82}$$

Equations of motion for the graviton.

In order to give the equations of motion for the graviton we need the expression of the supercovariant Ricci tensor, $R_{db} = R_{dcba}\eta^{ca}$, which is given by the superspace Bianchi identities at canonical dimension 2 (F.5). The corresponding Ricci scalar, $R = R_{db}\eta^{db}$, is then

$$R = -2\mathcal{D}^a\phi\mathcal{D}_a\phi - \frac{1}{2}H^{*a}H^*{}_a e^{-4\phi} + \frac{3}{4}e^{-2\phi}H^{*a}(\lambda^{\mathrm{A}}\sigma_a\bar{\lambda}_{\mathrm{A}}) + \frac{3}{8}(\lambda^{\mathrm{B}}\lambda^{\mathrm{A}})(\bar{\lambda}_{\mathrm{B}}\bar{\lambda}_{\mathrm{A}}). \tag{4.83}$$

The knowledge of these ingredients allows us to write down the Einstein equation

$$\begin{aligned} R_{db} - \frac{1}{2}\eta_{db}R &= -2\left[\mathcal{D}_d\phi\,\mathcal{D}_b\phi - \frac{1}{2}\eta_{db}\mathcal{D}^a\phi\mathcal{D}_a\phi\right] \\ &\quad -\frac{1}{2}e^{-4\phi}\left[H^*_d\,H^*_b - \frac{1}{2}\eta_{db}H^{*a}H^*{}_a\right] \\ &\quad -e^{-2\phi}\left[F_{df[\mathrm{BA}]}F_b{}^{f[\mathrm{BA}]} - \frac{1}{4}\eta_{db}F_{ef[\mathrm{BA}]}F^{ef[\mathrm{BA}]}\right] \\ &\quad -\frac{i}{8}\sum_{db}\left[\lambda^{\mathrm{F}}\sigma_d\mathcal{D}_b\bar{\lambda}_{\mathrm{F}} - (\mathcal{D}_b\lambda^{\mathrm{F}})\sigma_d\bar{\lambda}_{\mathrm{F}}\right] \\ &\quad -\frac{1}{8}\left[\frac{1}{4}(\lambda^{\mathrm{F}}\sigma_d\bar{\lambda}_{\mathrm{F}})\,(\lambda^{\mathrm{A}}\sigma_b\bar{\lambda}_{\mathrm{A}}) + \eta_{db}(\lambda^{\mathrm{B}}\lambda^{\mathrm{A}})(\bar{\lambda}_{\mathrm{B}}\bar{\lambda}_{\mathrm{A}})\right] \\ &\quad -\frac{1}{8}e^{-2\phi}\left[H^*_d(\lambda^{\mathrm{F}}\sigma_b\bar{\lambda}_{\mathrm{F}}) - 3\eta_{db}H^{*a}(\lambda^{\mathrm{A}}\sigma_a\bar{\lambda}_{\mathrm{A}})\right]\,, \end{aligned} \tag{4.84}$$

where one may recognize on the right-hand-side the usual terms of the energy-momentum tensor corresponding to matter fields: scalar fields, antisymmetric tensor, photon fields and spinor fields respectively. As it will be shown by (4.96), the contribution of the gravitini is hidden in R_{db}.

4.5 Equations of motion in terms of component fields

In the previous section we calculated the equations of motion for all component fields of the N-T multiplet (graviton (4.84), gravitini (4.81), (4.82), graviphotons (4.77), 1/2-spin fields (4.66), (4.67), scalar (4.72) and the antisymmetric tensor (4.74)) in terms of supercovariant objects, which have only flat (Lorentz) indices. In order to write these equations of motion in terms of component fields, one passes to curved (Einstein) indices by the standard way [71]. General formulae are easily written using the notation $E^{\mathcal{A}}|\!| = e^{\mathcal{A}} = dx^\mu e_\mu{}^{\mathcal{A}}$ [72].

4.5.1 Supercovariant→component toolkit

Recall that the graviton, gravitini and graviphotons are identified in the super-vielbein. Thus, their field strengths can be found in their covariant counterparts using

$$T^{\mathcal{A}}|\!| = \frac{1}{2}dx^\mu dx^\nu\left(\mathcal{D}_\nu e_\mu{}^{\mathcal{A}} - \mathcal{D}_\mu e_\nu{}^{\mathcal{A}}\right) = \frac{1}{2}e^{\mathcal{B}}e^{\mathcal{C}}T_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}}|\,. \tag{4.85}$$

For $\mathcal{A} = a$ one finds the relation

$$\mathcal{D}_\nu e_\mu{}^a - \mathcal{D}_\mu e_\nu{}^a \;=\; i\psi_{[\nu \mathrm{A}}\sigma^a\bar{\psi}_{\mu]}{}^{\mathrm{A}} , \tag{4.86}$$

which determinates the Lorentz connection in terms of the vierbein, its derivatives and gravitini fields. For $\mathcal{A} =^{\mathrm{A}}_{\alpha}$ and $\mathcal{A} =^{\dot\alpha}_{\mathrm{A}}$ we have the expression of the covariant field strength of the gravitini

$$\begin{aligned} T_{cb}{}^{\alpha}_{\mathrm{A}}| \;&=\; e_b{}^\mu e_c{}^\nu \mathcal{D}_{[\nu}\psi_{\mu]}{}^{\alpha}_{\mathrm{A}} - e_b{}^\mu e_c{}^\nu \frac{q}{4}\varepsilon_{\mathrm{DCBA}}\bar{\psi}_\nu{}^{\mathrm{D}}\bar{\psi}_\mu{}^{\mathrm{C}}\lambda^{\alpha\mathrm{B}} - \frac{i}{2}e_{[c}{}^\nu(\bar{\psi}_\nu{}^{\mathrm{B}}\bar{\sigma}_{b]}\sigma^{da})^\alpha F_{da[\mathrm{BA}]}|e^{-\phi} \\ &+\frac{i}{4}(\psi_{\nu\mathrm{F}}\sigma_{f[b})^\alpha e_{c]}{}^\nu \left[\lambda^{\mathrm{F}}\sigma^f\bar{\lambda}_{\mathrm{A}} - \frac{1}{2}\delta^{\mathrm{F}}_{\mathrm{A}}\lambda^{\mathrm{B}}\sigma^f\bar{\lambda}_{\mathrm{B}}\right] , \end{aligned} \tag{4.87}$$

$$\begin{aligned} T_{cb}{}^{\mathrm{A}}_{\dot\alpha}| \;&=\; e_b{}^\mu e_c{}^\nu \mathcal{D}_{[\nu}\bar{\psi}_{\mu]}{}^{\mathrm{A}}_{\dot\alpha} - e_b{}^\mu e_c{}^\nu \frac{q}{4}\varepsilon^{\mathrm{DCBA}}\psi_{\nu\mathrm{D}}\psi_{\mu\mathrm{C}}\bar{\lambda}_{\dot\alpha\mathrm{B}} - \frac{i}{2}e_{[c}{}^\nu(\psi_{\nu\mathrm{B}}\sigma_{b]}\bar{\sigma}^{da})_{\dot\alpha}F_{da}{}^{[\mathrm{BA}]}|e^{-\phi} \\ &-\frac{i}{4}(\bar{\psi}_\nu{}^{\mathrm{F}}\bar{\sigma}_{f[b})_{\dot\alpha}e_{c]}{}^\nu \left[\lambda^{\mathrm{A}}\sigma^f\bar{\lambda}_{\mathrm{F}} - \frac{1}{2}\delta^{\mathrm{A}}_{\mathrm{F}}\lambda^{\mathrm{B}}\sigma^f\bar{\lambda}_{\mathrm{B}}\right] . \end{aligned} \tag{4.88}$$

As for $\mathcal{A} = \mathbf{u}$, the central charge indices, we obtain the covariant field strength of the graviphotons

$$\begin{aligned} F_{ba}{}^{\mathbf{u}}| \;&=\; e_b{}^\nu e_a{}^\mu \mathcal{F}_{\nu\mu}{}^{\mathbf{u}} + e_b{}^\nu e_a{}^\mu \left[\bar{\psi}_\nu{}^{\mathrm{C}}\bar{\psi}_\mu{}^{\mathrm{B}} + i\bar{\psi}_{[\nu}{}^{\mathrm{C}}\bar{\sigma}_{\mu]}\lambda^{\mathrm{B}}\right] e^\phi \mathfrak{t}_{[\mathrm{CB}]}{}^{\mathbf{u}} \\ &+e_b{}^\nu e_a{}^\mu \left[\psi_{\nu\mathrm{C}}\psi_{\mu\mathrm{B}} + i\psi_{[\nu\mathrm{C}}\sigma_{\mu]}\bar{\lambda}_{\mathrm{B}}\right] e^\phi \mathfrak{t}^{[\mathrm{CB}]\mathbf{u}} , \end{aligned} \tag{4.89}$$

with $\mathcal{F}_{\nu\mu}{}^{\mathbf{u}}$ the field strength of the graviphotons $\mathcal{F}_{\nu\mu}{}^{\mathbf{u}} = \partial_\nu v_\mu{}^{\mathbf{u}} - \partial_\mu v_\nu{}^{\mathbf{u}}$. In the $SU(4)$ basis this becomes

$$\begin{aligned} F_{ba}{}^{[\mathrm{BA}]}| \;&=\; e_b{}^\nu e_a{}^\mu \mathcal{F}_{\nu\mu}{}^{[\mathrm{BA}]} + e_b{}^\nu e_a{}^\mu \left[\bar{\psi}_\nu{}^{[\mathrm{B}}\bar{\psi}_\mu{}^{\mathrm{A}]} + i\bar{\psi}_{[\nu}{}^{[\mathrm{B}}\bar{\sigma}_{\mu]}\lambda^{\mathrm{A}]}\right] e^\phi \\ &+e_b{}^\nu e_a{}^\mu \frac{q}{2}\varepsilon^{\mathrm{DCBA}} \left[\psi_{\nu\mathrm{D}}\psi_{\mu\mathrm{C}} + i\psi_{[\nu\mathrm{D}}\sigma_{\mu]}\bar{\lambda}_{\mathrm{C}}\right] e^\phi , \end{aligned} \tag{4.90}$$

with the field strength $\mathcal{F}_{\nu\mu}{}^{[\mathrm{BA}]} = \mathcal{F}_{\nu\mu}{}^{\mathbf{u}}\mathfrak{h}_{\mathbf{u}}{}^{[\mathrm{BA}]} = \partial_\nu V_\mu{}^{[\mathrm{BA}]} - \partial_\mu V_\nu{}^{[\mathrm{BA}]}$.

Since the antisymmetric tensor is identified in the 2–form, the development of its covariant field strength on component fields is deduced using

$$H|| \;=\; \frac{1}{2}dx^\mu dx^\nu dx^\rho \partial_\rho b_{\nu\mu} \;=\; \frac{1}{3!}e^{\mathcal{A}}e^{\mathcal{B}}e^{\mathcal{C}}H_{\mathcal{CBA}}| \tag{4.91}$$

and one finds

$$H^{*a}| \;=\; e_\lambda{}^a\mathcal{G}^\lambda + ie_\lambda{}^a\left[\psi_{\rho\mathrm{F}}\sigma^{\lambda\rho}\lambda^{\mathrm{F}} - \bar{\psi}_\rho{}^{\mathrm{F}}\bar{\sigma}^{\lambda\rho}\bar{\lambda}_{\mathrm{F}} + \frac{1}{2}\varepsilon^{\lambda\rho\nu\mu}\psi_{\rho\mathrm{F}}\sigma_\nu\bar{\psi}_\mu{}^{\mathrm{F}}\right] e^{2\phi} , \tag{4.92}$$

with

$$\mathcal{G}^\lambda \;=\; \frac{1}{2}\varepsilon^{\lambda\rho\nu\mu}\left[\partial_\rho b_{\nu\mu} - v_\rho{}^{\mathbf{u}}\mathfrak{g}_{\mathbf{uv}}\mathcal{F}_{\nu\mu}{}^{\mathbf{v}}\right] \;=\; \frac{1}{2}\varepsilon^{\lambda\rho\nu\mu}\left[\partial_\rho b_{\nu\mu} - V_{\rho[\mathrm{BA}]}\mathcal{F}_{\nu\mu}{}^{[\mathrm{BA}]}\right] . \tag{4.93}$$

Note, that the dual field strength, $\frac{1}{2}\varepsilon^{\lambda\rho\nu\mu}\partial_\rho b_{\nu\mu}$, of the antisymmetric tensor appears in company with the Chern-Simons term $\frac{1}{2}\varepsilon^{\lambda\rho\nu\mu}v_\rho{}^{\mathbf{u}}\mathfrak{g}_{\mathbf{uv}}\mathcal{F}_{\nu\mu}{}^{\mathbf{v}}$. We use the notation $\mathcal{G}^\lambda$ in order to accentuate this feature. Recall also, that one of the fundamental aims of the article [13] was to explain in detail that this phenomenon is quite general and arises as an intrinsic property of soldering in superspace with central charge coordinates.

The lowest component of the derivative of the scalar can be calculated using $D\phi|| = dx^\mu\mathcal{D}_\mu\phi = e^{\mathcal{A}}\mathcal{D}_{\mathcal{A}}\phi|$, and it is

$$\mathcal{D}_a\phi| = e_a{}^\mu\left(\mathcal{D}_\mu\phi - \frac{1}{4}\psi_{\mu\mathrm{F}}\lambda^{\mathrm{F}} - \frac{1}{4}\bar{\psi}_\mu{}^{\mathrm{F}}\bar{\lambda}_{\mathrm{F}}\right) , \tag{4.94}$$

while the lowest component of the double derivative $\mathcal{D}_a\mathcal{D}^a\phi|$, needed for the expansion of the equation of motion for the scalar (4.72), becomes

$$\begin{aligned}
2\mathcal{D}_a\mathcal{D}^a\phi| &= 2\Box\phi + e_a{}^\mu\mathcal{D}_\mu e^{a\nu}\left[2\mathcal{D}_\nu\phi - \frac{1}{2}\psi_{\nu\mathrm{F}}\lambda^{\mathrm{F}} - \frac{1}{2}\bar\psi_\nu{}^{\mathrm{F}}\bar\lambda_{\mathrm{F}}\right] - \frac{1}{2}H^{*a}|\psi_{\mu\mathrm{F}}\sigma_a\bar\psi^{\mu\mathrm{F}}\,e^{-2\phi} \\
&\quad -\frac{1}{2}\mathcal{D}^\mu(\psi_{\mu\mathrm{F}}\lambda^{\mathrm{F}} + \bar\psi_\mu{}^{\mathrm{F}}\bar\lambda_{\mathrm{F}}) - \frac{1}{2}(\psi_{\mu\mathrm{F}}\mathcal{D}^\mu\lambda^{\mathrm{F}} + \bar\psi_\mu{}^{\mathrm{F}}\mathcal{D}^\mu\bar\lambda_{\mathrm{F}}) \\
&\quad -\frac{3i}{32}(\lambda^{\mathrm{C}}\sigma^\nu\bar\lambda_{\mathrm{C}})\left(\psi_{\nu\mathrm{F}}\lambda^{\mathrm{F}} - \bar\psi_\nu{}^{\mathrm{F}}\bar\lambda_{\mathrm{F}}\right) - \frac{3}{4}(\psi_{\mu\mathrm{F}}\lambda^{\mathrm{A}})(\bar\psi_\mu{}^{\mathrm{F}}\bar\lambda_{\mathrm{A}}) \\
&\quad -\frac{1}{4}(\psi_{\mu\mathrm{F}}\lambda^{\mathrm{F}})(\psi^\mu_{\mathrm{C}}\lambda^{\mathrm{C}}) + \frac{1}{2}(\psi^\mu_{\mathrm{F}}\lambda^{\mathrm{F}})(\bar\psi_\mu{}^{\mathrm{C}}\bar\lambda_{\mathrm{C}}) - \frac{1}{4}(\bar\psi_\mu{}^{\mathrm{F}}\bar\lambda_{\mathrm{F}})(\bar\psi^{\mu\mathrm{C}}\bar\lambda_{\mathrm{C}}) \\
&\quad +F_{ba[\mathrm{FC}]}|\left[\frac{1}{2}\bar\psi_\mu{}^{\mathrm{F}}\bar\sigma^{ba}\bar\psi^{\mu\mathrm{C}} + \frac{i}{4}\bar\psi_\nu{}^{\mathrm{F}}\bar\sigma^\nu\sigma^{ba}\lambda^{\mathrm{C}}\right]e^{-\phi} \\
&\quad +F_{ba}{}^{[\mathrm{FC}]}|\left[\frac{1}{2}\psi_{\mu\mathrm{F}}\sigma^{ba}\psi^\mu_{\mathrm{C}} + \frac{i}{4}\psi_{\nu\mathrm{F}}\sigma^\nu\bar\sigma^{ba}\bar\lambda_{\mathrm{C}}\right]e^{-\phi}\,. \qquad (4.95)
\end{aligned}$$

In order to compare our results with the component expression of the scalar's equation of motion derived from [12], we have to replace in this expression $e_a{}^\mu\mathcal{D}_\mu e^{a\nu}$ with

$$e_a{}^\mu\mathcal{D}_\mu e^{a\nu} = V^{-1}\partial_\mu(Vg^{\mu\nu}) - ig^{\mu\nu}\psi_{[\mu\mathrm{A}}\sigma^k\bar\psi_{k]}{}^{\mathrm{A}}\,,$$

as a consequence of (4.86).

Finally, using $R_b{}^a\| = \frac{1}{2}dx^\mu dx^\nu\mathcal{R}_{\nu\mu b}{}^a = \frac{1}{2}e^{\mathcal{B}}e^{\mathcal{C}}R_{\mathcal{CB}b}{}^a|$, one obtains for the lowest component of the covariant Ricci tensor R_{db} the expression

$$\begin{aligned}
R_{db}| &= \frac{1}{2}\sum_{db}\Big\{e_d{}^\nu e^{\mu a}\mathcal{R}_{\nu\mu ba}| + \frac{1}{2}\varepsilon_b{}^{\mu ef}\psi_{\mu\mathrm{D}}\sigma_d T_{ef}{}^{\mathrm{D}}| - \frac{1}{2}\varepsilon_b{}^{\mu ef}\bar\psi_\mu{}^{\mathrm{D}}\bar\sigma_d T_{ef\,\mathrm{D}}| \\
&\quad +\frac{1}{4}\left(i\psi^\mu_{\mathrm{D}}\sigma_d\bar\sigma^f\sigma_{b\mu}\lambda^{\mathrm{D}} - ie_d{}^\nu\delta^f_b\psi_{\nu\mathrm{D}}\lambda^{\mathrm{D}}\right)P_f| \\
&\quad +\frac{1}{4}\left(i\bar\psi^{\mu\mathrm{D}}\bar\sigma_d\sigma^f\bar\sigma_{b\mu}\bar\lambda_{\mathrm{D}} - ie_d{}^\nu\delta^f_b\bar\psi^{\mathrm{D}}_\nu\bar\lambda_{\mathrm{D}}\right)\bar P_f| \\
&\quad -\frac{1}{2}e^{-\phi}F^{ef[\mathrm{DF}]}|\left(i\mathrm{tr}(\sigma_{b\mu}\sigma_{ef})\psi^\mu_{\mathrm{D}}\sigma_d\bar\lambda_{\mathrm{F}} + \frac{i}{2}e_d{}^\nu\psi_{\nu\mathrm{D}}\sigma_{ef}\sigma_b\bar\lambda_{\mathrm{F}}\right) \\
&\quad -\frac{1}{2}e^{-\phi}F^{ef}{}_{[\mathrm{DF}]}|\left(i\mathrm{tr}(\bar\sigma_{b\mu}\bar\sigma_{ef})\bar\psi^{\mu\mathrm{D}}\bar\sigma_d\lambda^{\mathrm{F}} + \frac{i}{2}e_d{}^\nu\bar\psi^{\mathrm{D}}_\nu\bar\sigma_{ef}\bar\sigma_b\lambda^{\mathrm{F}}\right) \\
&\quad +\frac{1}{2}e_d{}^\nu e^{-\phi}\left(\mathrm{tr}(\bar\sigma_b{}^\mu\bar\sigma_{ef})\psi_{\nu\mathrm{D}}\psi_{\mu\mathrm{C}}F^{ef[\mathrm{DC}]}| + \mathrm{tr}(\sigma_b{}^\mu\sigma_{ef})\bar\psi^{\mathrm{D}}_\nu\bar\psi^{\mathrm{C}}_\mu F^{ef}{}_{[\mathrm{DC}]}|\right) \\
&\quad -\frac{1}{2}e_d{}^\mu(\delta^{\mathrm{D}}_{\mathrm{B}}\delta^{\mathrm{C}}_{\mathrm{A}} - \frac{1}{2}\delta^{\mathrm{D}}_{\mathrm{A}}\delta^{\mathrm{C}}_{\mathrm{B}})\left[(\psi_{[\mu\mathrm{D}}\sigma_b{}^\nu\lambda^{\mathrm{B}})(\bar\psi_{\nu]}{}^{\mathrm{A}}\bar\lambda_{\mathrm{B}}) - (\psi_{[\mu\mathrm{D}}\lambda^{\mathrm{B}})(\bar\psi_{\nu]}{}^{\mathrm{A}}\bar\sigma_b{}^\nu\bar\lambda_{\mathrm{B}})\right]\Big\}\,. \qquad (4.96)
\end{aligned}$$

4.5.2 The equations of motion

In the last subsection we deduced the expression of all quantities appearing in the supercovariant equations of motion in terms of component fields. We are therefore ready now to replace these expressions in (4.84), (4.81), (4.82), (4.77), (4.66), (4.67), (4.72), (4.74) and give the equations of motion in terms of component fields.

It turns out that the expressions

$$\begin{aligned}\tilde{H}_\rho &= e_\rho{}^a H_a^*| - \frac{i}{2}e^{2\phi}\psi_{\rho\mathrm{A}}\lambda^{\mathrm{A}} + \frac{i}{2}e^{2\phi}\bar{\psi}_\rho{}^{\mathrm{A}}\bar{\lambda}_{\mathrm{A}} - \frac{3}{4}e^{2\phi}\lambda^{\mathrm{A}}\sigma_\rho\bar{\lambda}_{\mathrm{A}} \\ &= \mathcal{G}_\rho + \frac{i}{2}e^{2\phi}\left[\psi_{\kappa\mathrm{F}}\sigma_\rho\bar{\sigma}^\kappa\lambda^{\mathrm{F}} - \bar{\psi}_\kappa{}^{\mathrm{F}}\bar{\sigma}_\rho\sigma^\kappa\bar{\lambda}_{\mathrm{F}} + \varepsilon_\rho{}^{\kappa\nu\mu}\psi_{\kappa\mathrm{F}}\sigma_\nu\bar{\psi}_\mu{}^{\mathrm{F}}\right] - \frac{3}{4}e^{2\phi}\lambda^{\mathrm{A}}\sigma_\rho\bar{\lambda}_{\mathrm{A}} \end{aligned} \tag{4.97}$$

and

$$\begin{aligned}\tilde{F}^{\nu\kappa\mathbf{z}} &= e^{\nu b}e^{\kappa a}F_{ba}{}^{\mathbf{z}}| + \frac{i}{2}\varepsilon^{\nu\kappa\mu\rho}\left[\bar{\psi}_\mu{}^{\mathrm{D}}\bar{\psi}_\rho{}^{\mathrm{C}} - i\frac{q}{2}\varepsilon^{\mathrm{DCBA}}\bar{\lambda}_{\mathrm{B}}\bar{\sigma}_\mu\psi_{\rho\mathrm{A}}\right]e^\phi t_{[\mathrm{DC}]}{}^{\mathbf{z}} \\ &\quad -\frac{i}{2}\varepsilon^{\nu\kappa\mu\rho}\left[\psi_{\mu\mathrm{D}}\psi_{\rho\mathrm{C}} - i\frac{q}{2}\varepsilon_{\mathrm{DCBA}}\lambda^{\mathrm{B}}\sigma_\mu\bar{\psi}_\rho{}^{\mathrm{A}}\right]e^\phi t^{[\mathrm{DC}]\mathbf{z}} \\ &= \mathcal{F}^{\nu\kappa\mathbf{z}} - \mathrm{tr}(\sigma^{\nu\kappa}\sigma^{\mu\rho})\left[\bar{\psi}_\mu{}^{\mathrm{D}}\bar{\psi}_\rho{}^{\mathrm{C}} - i\frac{q}{2}\varepsilon^{\mathrm{DCBA}}\bar{\lambda}_{\mathrm{B}}\bar{\sigma}_\mu\psi_{\rho\mathrm{A}}\right]e^\phi t_{[\mathrm{DC}]}{}^{\mathbf{z}} \\ &\quad -\mathrm{tr}(\bar{\sigma}^{\nu\kappa}\bar{\sigma}^{\mu\rho})\left[\psi_{\mu\mathrm{D}}\psi_{\rho\mathrm{C}} - i\frac{q}{2}\varepsilon_{\mathrm{DCBA}}\lambda^{\mathrm{B}}\sigma_\mu\bar{\psi}_\rho{}^{\mathrm{A}}\right]e^\phi t^{[\mathrm{DC}]\mathbf{z}} \end{aligned} \tag{4.98}$$

appear systematically, and using them, the equations take a quite simple form. Let us also denote the quantity $\hat{F}_{\nu\kappa}{}^{\mathbf{z}} = e_\nu{}^b e_\kappa{}^a F_{ba}{}^{\mathbf{z}}|$, which is called the supercovariant field strength of the graviphotons in the component approach [88], [12].

Equations of motion for the helicity 1/2 fields.

$$\begin{aligned}\left(\sigma^\mu\hat{\mathcal{D}}_\mu\bar{\lambda}_{\mathrm{A}}\right)_\beta &= -ie^{-2\phi}\tilde{H}^\mu\left[\frac{i}{2}(\sigma^\nu\bar{\sigma}_\mu\psi_{\nu\mathrm{A}})_\beta - \frac{3}{4}(\sigma_\mu\bar{\lambda}_{\mathrm{A}})_\beta\right] - \frac{i}{2}(\bar{\psi}_\nu{}^{\mathrm{F}}\bar{\lambda}_{\mathrm{F}})(\sigma^\mu\bar{\sigma}^\nu\psi_{\mu\mathrm{A}})_\beta \\ &\quad +i\partial_\nu\phi(\sigma^\mu\bar{\sigma}^\nu\psi_{\mu\mathrm{A}})_\beta - e^{-\phi}\hat{F}_{\kappa\rho[\mathrm{FA}]}(\sigma^\mu\bar{\sigma}^{\kappa\rho}\bar{\psi}_\mu{}^{\mathrm{F}})_\beta - \frac{3i}{8}(\bar{\lambda}_{\mathrm{A}}\bar{\lambda}_{\mathrm{F}})\lambda_\beta^{\mathrm{F}} \end{aligned} \tag{4.99}$$

Equations of motion for the gravitini.

$$\begin{aligned}\varepsilon^{\rho\kappa\nu\mu}(\bar{\sigma}_\kappa\hat{\mathcal{D}}_\nu\psi_{\mu\mathrm{A}})^{\dot{\alpha}} &= -\frac{i}{4}e^{-2\phi}\tilde{H}_\nu\left[\varepsilon^{\rho\kappa\nu\mu}(\bar{\sigma}_\kappa\psi_{\mu\mathrm{A}})^{\dot{\alpha}} + (\bar{\lambda}_{\mathrm{A}}\bar{\sigma}^\rho\sigma^\nu\varepsilon)^{\dot{\alpha}}\right] - \frac{1}{2}\partial_\nu\phi(\bar{\lambda}_{\mathrm{A}}\bar{\sigma}^\rho\sigma^\nu\varepsilon)^{\dot{\alpha}} \\ &\quad -e^{-\phi}\hat{F}_{\mu\nu[\mathrm{AF}]}\left[\mathrm{tr}(\sigma^{\rho\kappa}\sigma^{\mu\nu})\bar{\psi}_\kappa{}^{\mathrm{F}\dot{\alpha}} + \frac{i}{2}\mathrm{tr}(\bar{\sigma}^{\rho\kappa}\bar{\sigma}^{\mu\nu})(\sigma_\kappa\lambda^{\mathrm{F}})^{\dot{\alpha}}\right] \\ &\quad +\frac{1}{8}\psi_{\nu\mathrm{A}}\lambda^{\mathrm{F}}(\bar{\sigma}^{\rho\nu}\bar{\lambda}_{\mathrm{F}})^{\dot{\alpha}} + \frac{3}{8}(\psi_{\nu\mathrm{A}}\sigma^{\rho\nu}\lambda^{\mathrm{F}})\bar{\lambda}_{\mathrm{F}}^{\dot{\alpha}} - \frac{1}{4}(\psi_{\nu\mathrm{F}}\sigma^\rho\bar{\sigma}^\nu\lambda^{\mathrm{F}})\bar{\lambda}_{\mathrm{A}}^{\dot{\alpha}} \\ &\quad +\frac{q}{4}\varepsilon^{\rho\kappa\nu\mu}\varepsilon_{\mathrm{CBFA}}\bar{\psi}_\nu{}^{\mathrm{C}}\bar{\psi}_\mu{}^{\mathrm{B}}(\bar{\sigma}_\kappa\lambda^{\mathrm{F}})^{\dot{\alpha}} \end{aligned} \tag{4.100}$$

Equations of motion for the scalar.

$$\begin{aligned} 0 &= 2V^{-1}\partial_\mu(Vg^{\mu\nu}\partial_\nu\phi) + \frac{1}{2}V^{-1}\partial_\mu\left(V\lambda^{\mathrm{A}}\sigma^\nu\bar{\sigma}^\mu\psi_{\nu\mathrm{A}} + V\bar{\lambda}_{\mathrm{A}}\bar{\sigma}^\nu\sigma^\mu\bar{\psi}_\nu^{\mathrm{A}}\right) \\ &\quad -e^{-4\phi}\mathcal{G}^\mu\tilde{H}_\mu + \frac{1}{2}e^{-2\phi}\mathcal{F}_{\nu\mu[\mathrm{BA}]}\tilde{F}^{\nu\mu[\mathrm{BA}]} \end{aligned} \tag{4.101}$$

Equations of motion for the antisymmetric tensor.

$$\partial_\kappa\left(e^{-4\phi}V\varepsilon^{\mu\nu\kappa\rho}\tilde{H}_\rho\right) = 0 \tag{4.102}$$

Equations of motion for the graviphotons.

$$\partial_\nu \left(V e^{-2\phi} \tilde{F}^{\nu\kappa\mathbf{u}} \right) = \frac{1}{2} V e^{-4\phi} \varepsilon^{\rho\nu\mu\kappa} \tilde{H}_\rho \mathcal{F}_{\nu\mu}{}^{\mathbf{u}} \tag{4.103}$$

Equations of motion for the graviton.

The Einstein equation in terms of component fields is also deduced in a straightforward manner from (4.84) and (4.96) with the usual Ricci tensor $\mathcal{R}_{\mu\nu} = \frac{1}{2} \sum_{\mu\nu} e_\nu{}^b e^{\kappa a} \mathcal{R}_{\mu\kappa ba}$. Here we give the expression of the Ricci scalar:

$$\begin{aligned}
\mathcal{R} = {} & \frac{1}{2} \varepsilon^{\rho\kappa\nu\mu} \psi_{\rho\mathrm{A}} \sigma_\kappa \hat{\mathcal{D}}_\nu \bar{\psi}_\mu{}^{\mathrm{A}} - \frac{1}{2} \varepsilon^{\rho\kappa\nu\mu} \bar{\psi}_\rho{}^{\mathrm{A}} \bar{\sigma}_\kappa \hat{\mathcal{D}}_\nu \psi_{\mu\mathrm{A}} \\
& - \frac{i}{4} \lambda^{\mathrm{A}} \sigma^\mu \hat{\mathcal{D}}_\mu \bar{\lambda}_{\mathrm{A}} - \frac{i}{4} \bar{\lambda}_{\mathrm{A}} \bar{\sigma}^\mu \hat{\mathcal{D}}_\mu \lambda^{\mathrm{A}} - 2 \partial^\mu \phi \partial_\mu \phi \\
& - e^{-\phi} \hat{F}_{\kappa\rho}{}^{[\mathrm{BA}]} \left(\mathrm{tr}(\bar{\sigma}^{\kappa\rho} \bar{\sigma}^{\mu\nu}) \psi_{\mu\mathrm{B}} \psi_{\nu\mathrm{A}} + \frac{i}{2} \mathrm{tr}(\sigma^{\kappa\rho} \sigma^{\mu\nu}) \psi_{\mu\mathrm{B}} \sigma_\nu \bar{\lambda}_{\mathrm{A}} \right) \\
& - e^{-\phi} \hat{F}_{\kappa\rho[\mathrm{BA}]} \left(\mathrm{tr}(\sigma^{\kappa\rho} \sigma^{\mu\nu}) \bar{\psi}_\mu{}^{\mathrm{B}} \bar{\psi}_\nu{}^{\mathrm{A}} + \frac{i}{2} \mathrm{tr}(\bar{\sigma}^{\kappa\rho} \bar{\sigma}^{\mu\nu}) \bar{\psi}_\mu{}^{\mathrm{B}} \bar{\sigma}_\nu \lambda^{\mathrm{A}} \right) \\
& - \frac{1}{2} e^{-4\phi} \tilde{H}^\rho \left(\mathcal{G}_\rho + \frac{i}{2} e^{2\phi} (\psi_{\kappa\mathrm{F}} \sigma_\rho \bar{\sigma}^\kappa \lambda^{\mathrm{F}} - \bar{\psi}_\kappa{}^{\mathrm{F}} \bar{\sigma}_\rho \sigma^\kappa \bar{\lambda}_{\mathrm{F}} + 2 \varepsilon_\rho{}^{\kappa\nu\mu} \psi_{\kappa\mathrm{F}} \sigma_\nu \bar{\psi}_\mu{}^{\mathrm{F}} \right) \\
& + \frac{1}{2} (\psi_{\rho\mathrm{A}} \lambda^{\mathrm{A}}) (\bar{\psi}^{\rho\mathrm{A}} \bar{\lambda}_{\mathrm{A}}) + \frac{i}{2} (\lambda^{\mathrm{F}} \sigma^\rho \bar{\lambda}_{\mathrm{F}}) (\psi_{\rho\mathrm{A}} \lambda^{\mathrm{A}} - \bar{\psi}_\rho{}^{\mathrm{A}} \bar{\lambda}_{\mathrm{A}}) \\
& - \frac{3i}{16} \varepsilon^{\rho\mu\nu\kappa} (\psi_{\rho\mathrm{F}} \sigma_\mu \bar{\psi}_\nu{}^{\mathrm{F}}) (\lambda^{\mathrm{A}} \sigma_\kappa \bar{\lambda}_{\mathrm{A}}) - \frac{i}{2} \varepsilon^{\rho\mu\nu\kappa} (\psi_{\rho\mathrm{F}} \sigma_\mu \bar{\psi}_\nu{}^{\mathrm{A}}) (\lambda^{\mathrm{F}} \sigma_\kappa \bar{\lambda}_{\mathrm{A}})
\end{aligned} \tag{4.104}$$

4.6 Conclusion

The aim of this work was to deduce the equations of motion for the components of the N-T multiplet from its geometrical description in central charge superspace, and compare these equations with those deduced from the Lagrangian of the component formulation of the theory with the same field content [12].

We showed that the constraints on the superspace which allow to identify the components in the geometry imply equations of motion in terms of supercovariant quantities. Moreover, we succeeded in writing these equations of motion in terms of component fields in an elegant way, using the objects $\tilde{H}_\mu$ and $\tilde{F}_{\mu\nu}{}^{\mathbf{u}}$. The equations found this way are in perfect concordance with the ones deduced from the Lagrangian of Nicolai and Townsend [12]. This result resolves all remaining doubt about the equivalence of the geometric description on central charge superspace of the N-T multiplet and the Lagrangian formulation of the theory with the same field content.

As a completion of this work one may ask oneself about an interpretation of the objects $\tilde{H}_\mu$ and $\tilde{F}_{\mu\nu}{}^{\mathbf{u}}$, which seem to be some natural building blocks of the Lagrangian. Concerning this question let us just remark the simplicity of the relation

$$-i\chi_\mu{}^{\mathrm{A}}{}_{\mathrm{A}} = e^{-2\phi} \tilde{H}_\mu + \frac{3}{8} \lambda^{\mathrm{A}} \sigma_\mu \bar{\lambda}_{\mathrm{A}} \tag{4.105}$$

between $\tilde{H}_\mu$ and the $U(1)$ part of the initial connection (4.45) of the central charge superspace with structure group $SL(2, \mathbb{C}) \otimes U(4)$.

Having at our disposal now a well-defined and elegant formalism to describe $N = 4$ supergravity, it would be interesting to try to incorporate matter multiplets in this framework. In particular, according to [11], 6 vector multiplets naturally couple to gravity in the process of dimensional reduction. In the central charge superspace we have used here, before putting all

constraints, the maximal number of central charges is 12. Once the constraints have been taken into account, this number reduces to 6 (4.31), which is exactly the number of vectors contained in the $N = 4$ gravity multiplet. However, if one relaxes some of the constraints listed in this chapter, one may expect that the full number of supercharges would be available. This would mean that 6 new vectors appear, and it seems natural to think that these vectors would correspond to those arising in the reduction of the gravity multiplet of $N = 1$ $d = 10$ supergravity [11]. We hope to report on this issue in a close futur.

Appendix A

Notions of differential geometry

A.1 Einstein's Gravity

We take the signature $(-,+,+,\ldots)$. The space-time has d dimensions and indices are running from 0 to $d-1$. The Christoffel connection is the unique torsion-free connection for which the metric is covariantly constant

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \tag{A.1}$$

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma_{\mu\nu}{}^{\lambda} g_{\lambda\rho} - \Gamma_{\mu\rho}{}^{\lambda} g_{\nu\lambda} = 0. \tag{A.2}$$

Here is a useful relation :

$$\partial_\mu\left(\sqrt{-g}V^\mu\right) = \sqrt{-g}\nabla_\mu V^\mu \tag{A.3}$$

that can be checked using the formula for the determinant of a matrix A

$$\det(A) = exp\,(\mathrm{Tr}\ln A)\,. \tag{A.4}$$

Since there is no torsion, the commutator of two covariant derivatives is given by the Riemann tensor as follows

$$[\nabla_\mu\,,\,\nabla_\nu]V_\rho = -\mathcal{R}_{\mu\nu\rho}{}^{\lambda}V_\lambda \tag{A.5}$$

and its expression is

$$\mathcal{R}_{\mu\nu\rho}{}^{\lambda} = \partial_\mu\Gamma_{\nu\rho}{}^{\lambda} - \partial_\nu\Gamma_{\mu\rho}{}^{\lambda} + \Gamma_{\mu\sigma}{}^{\lambda}\Gamma_{\nu\rho}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\lambda}\Gamma_{\mu\rho}{}^{\sigma}. \tag{A.6}$$

The Ricci tensor and scalar are :

$$\begin{aligned} \mathcal{R}_{\mu\nu} &= \mathcal{R}_{\mu\lambda\nu}{}^{\lambda} \\ \mathcal{R} &= g^{\mu\nu}\mathcal{R}_{\mu\nu}. \end{aligned} \tag{A.7}$$

Under a Weyl rescaling

$$\begin{aligned} g_{\mu\nu} &= \Omega^{-2}\,\hat{g}_{\mu\nu} \\ g^{\mu\nu} &= \Omega^{2}\,\hat{g}^{\mu\nu} \end{aligned} \tag{A.8}$$

$$\sqrt{-g} = \Omega^{-d}\sqrt{-\hat{g}} \tag{A.9}$$

it behaves like

$$\int \mathrm{d}x^d \sqrt{-g}\,\Omega^{d-2}\mathcal{R} \quad = \quad \int \mathrm{d}x^d \sqrt{-\hat{g}} \left(\mathcal{R} + (d-1)(d-2)(\frac{\partial\Omega}{\Omega})^2 \right). \tag{A.10}$$

Finally, the covariant derivative acts on a Majorana spinor η_α with the connection Γ_{mab} as

$$\nabla_m \eta_\alpha = \partial_m \eta_\alpha - \frac{1}{4}\Gamma_{mab}(\Gamma^{ab})_\alpha{}^\beta \eta_\beta \tag{A.11}$$

where Γ^{ab} is the antisymmetrized product of two Γ-matrices.

A.2 Forms

A p-form is defined by :

$$F_p \quad = \quad \frac{1}{p!} F_{M_1 M_2 \dots M_p} \mathrm{d}y^{M_1} \mathrm{d}y^{M_2} \dots \mathrm{d}y^{M_p}. \tag{A.12}$$

The Hodge dual is

$$* F_p \quad = \quad \frac{1}{\sqrt{-g}} \frac{1}{p!(d-p)!} F_{M_1 M_2 \dots M_p} \epsilon^{M_1 M_2 \dots M_p}{}_{M_{p+1} \dots M_d} \mathrm{d}y^{M_{p+1}} \dots \mathrm{d}y^{M_d}. \tag{A.13}$$

where we use for the epsilon tensor the convention $\epsilon^{12\dots d} = +1$ and

$$\epsilon_{M_1 M_2 \dots M_d} \quad = \quad g_{M_1 N_1} \dots g_{M_d N_d} \epsilon^{N_1 \dots N_d}. \tag{A.14}$$

This tensor enjoys the following properties

$$\begin{aligned}
\epsilon^{M_1 \dots M_d} \epsilon_{M_1 \dots M_d} &= g \cdot d! \\
\epsilon^{M_1 M_2 \dots M_d} \epsilon_{N_1 M_2 \dots M_d} &= g \cdot (d-1)! \delta^{M_1}_{N_1} \\
\epsilon^{M_1 M_2 M_3 \dots M_d} \epsilon_{N_1 N_2 M_3 \dots M_d} &= g \cdot (d-2)! \delta^{M_1 M_2}_{N_1 N_2} \\
\epsilon^{M_1 \dots M_p \dots M_d} \epsilon_{N_1 \dots N_p M_{p+1} \dots M_d} &= g \cdot (d-p)! \delta^{M_1 \dots M_p}_{N_1 \dots N_p}.
\end{aligned} \tag{A.15}$$

where $g = \det(g_{MN})$ and the δ-symbols satisfy

$$\delta^{M_1 \dots M_p}_{N_1 \dots N_p} F_{M_1 \dots M_p} \quad = \quad p!\, F_{N_1 \dots N_p}. \tag{A.16}$$

Then we have that

$$\begin{aligned}
H_p * F_p &= -\frac{1}{p!} H^{M_1 \dots M_p} F_{M_1 \dots M_p} \sqrt{-g} d^d y \\
* * F_p &= -(-)^{p(d-p)} F_p,
\end{aligned} \tag{A.17}$$

the extra minus sign being due to the negative signature of the minkowsky space (absent for Euclidean manifolds). Suppose the whole space-time splits into $\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{I}_6$, where $\mathcal{I}_6$ is an internal manifold. Then the coordinates of both spaces do not mix, and it is possible to define

the Hodge dual on each space. In this particular case, the 10-dimensional Hodge dual splits in the product of 2 forms, one belonging to each space, in the following way

$$*_{10}(E_n \wedge I_p) = (-)^{np} *_4 E_n \wedge *_6 I_p, \tag{A.18}$$

where $*_4$ and $*_6$ are defined as in (A.13), E_n and I_p are an external n-form and an internal p-form.

Finally, the action of the differential operator d, which brings a p-form to a (p+1)-form, reads

$$dF_p = \frac{1}{p!}\partial_N F_{M_1\ldots M_p}\, dy^N dy^{M_1}\ldots dy^{M_p} \tag{A.19}$$

and satisfies

$$d(F_p G_q) = dF.G + (-)^p F.dG. \tag{A.20}$$

It is a nilpotent operator

$$ddF_p = 0 \tag{A.21}$$

and satisfies Stoke's theorem

$$\int_M dF_p = \int_{\partial M} F_p. \tag{A.22}$$

Its conjugate $d\dagger$ brings a p-form to a (p-1)-form. The generalized Laplacian is

$$\Delta = dd^\dagger + d^\dagger d, \tag{A.23}$$

and the component expression of these operators[1] is

$$d^\dagger F_p = -\frac{1}{(p-1)!}(-)^{(p-1)(d-p)}\nabla^\nu (F_p)_{\nu\nu_1\ldots\nu_{p-1}}\mathrm{d}x^{\nu_1}\ldots\mathrm{d}x^{\nu_{p-1}} \tag{A.24}$$

$$(\Delta F_p)_{\nu_1\ldots\nu_p} = -\nabla^\nu\nabla_\nu (F_p)_{\nu_1\ldots\nu_p} + \sum_i [\nabla^\nu\,,\,\nabla_{\nu_i}](F_p)_{\nu_1\ldots\nu_{(i)}\ldots\nu_p}. \tag{A.25}$$

Using the positivity of the scalar product (for Euclidean spaces)

$$< F_p|H_p >= \int F_p \wedge *H_p \tag{A.26}$$

and considering the product $< F_p|\Delta F_p >$, one can show that a form is harmonic if and only if it is closed and co-closed

$$\Delta F_p = 0 \Longleftrightarrow dF_p = 0 \quad \text{and} \quad d^\dagger F_p = 0. \tag{A.27}$$

[1] For Euclidean spaces, signs are opposite.

A.3 Clifford algebra in 6 Euclidean dimensions

For a review of these results, see [89]. We consider Majorana spinors. There are six Gamma matrices, of dimension 8 obeying Clifford algebra

$$\{\Gamma_m\,,\,\Gamma_n\} = 2\eta_{mn}. \tag{A.28}$$

Their conjugation relation is $\Gamma_m^\dagger = \Gamma_m$. The chirality operator Γ_7 is defined as

$$\Gamma_7 = i\Gamma_1 \dots \Gamma_6 \tag{A.29}$$

and satisfies the same conjugation relation $(\Gamma_7)^\dagger = \Gamma_7$. Majorana spinors on the 6 dimensional internal space can be defined if we adopt the following conventions for the charge conjugation matrix $\mathcal{C}$

$$\mathcal{C}^T = \mathcal{C}\ , \qquad\qquad \Gamma_m^T = -\mathcal{C}\Gamma_m\mathcal{C}^{-1}\ , \tag{A.30}$$

while the Majorana condition on a spinor η reads

$$\eta^\dagger = \eta^T\mathcal{C}. \tag{A.31}$$

From the single Γ-matrices, we can build the antisymmetrized product of k matrices

$$(\Gamma^{(k)})_{m_1 m_2 \dots m_k} = \frac{1}{k!}\sum_{\sigma\in S^k}\epsilon(\sigma)\Gamma_{m_{\sigma(1)}}\Gamma_{m_{\sigma(2)}}\dots\Gamma_{m_{\sigma(k)}}, \tag{A.32}$$

which is a basis for all 8-dimensional matrices for k from 0 to 6. For $k=0$ and $k=3$ (modulo 4), $\Gamma^{(k)}$ is symmetric[2], and for $k=1$ or $k=2$ it is antisymmetric. These symmetry properties of the gamma matrices and $\mathcal{C}$ with the above conventions imply that for a commuting Majorana spinor η the following quantities vanish

$$\eta^\dagger\Gamma_{(1)}\eta = \eta^\dagger\Gamma_{(2)}\eta = \eta^\dagger\Gamma_{(5)}\eta = \eta^\dagger\Gamma_{(6)}\eta = 0\ , \tag{A.33}$$

We will meet several times the product of two such matrices. As an 8-dimensional matrix, this product can be expanded on the $\Gamma^{(k)}$. The (heuristic) rule is the following. Take the product $\Gamma_{m_1\dots m_k}\Gamma^{n_1\dots n_p}$. It will be a linear combination of Γ^l for l from 0 to $k+p$. Since the only tensor we can use, other than the matrices, is the δ which has two indices, the only terms which survive are the ones of order $k+p$, $k+p-2$ and so on, until there are no ways to take indices from the lowest order matrix in the product. The expansion reads

$$\Gamma_{m_1\dots m_k}\Gamma^{n_1\dots n_p} = \Gamma_{m_1\dots m_k}{}^{n_1\dots n_p} + A\delta_{[m_1}{}^{[n_1}\Gamma_{m_2\dots m_k]}{}^{n_2\dots n_p]} + \dots \tag{A.34}$$

For the sign of A, look at the first term $\Gamma_{m_1\dots m_k}{}^{n_1\dots n_p}$ and imagine you take n_1 to the right of m_1. Then you have to go through $m_k, m_{k-1}\dots m_2$. Each time you get a minus sign, so the sign of A is $(-)^{k-1}$. Its absolute value is $C_k^1 C_p^1$, where the 1 stands for the number of indices you take in the Γ, and C is the combination $C_n^l = \frac{n!}{l!(n-l)!}$. The next term will have an order 2 δ, its sign will be the previous sign times the sign obtained when going with n_2 through $m_k, m_{k-1}\dots m_3$ to go to the right of m_2, so$(-)^{k-1}(-)^{k-2}$, and its coefficient will be $C_k^2 C_p^2$, and so on. Suppose $k<p$, then the last term in the expansion will be proportional to $\delta^{[n_1 n_2\dots n_k}_{m_1 m_2\dots m_k}\Gamma^{n_{k+1}\dots n_p]}$. The absolute value of its coefficient will be $C_k^k C_p^k$. Let's take an explicit example to make things clearer. Following the above rules, it can be checked that

$$\begin{aligned}\Gamma_{m_1 m_2 m_3}\Gamma^{n_1 n_2 n_3 n_4} \quad &= \quad \Gamma_{m_1 m_2 m_3}{}^{n_1 n_2 n_3 n_4} + 3*4\delta^{[n_1}_{[m_1}\Gamma_{m_2 m_3]}{}^{n_2 n_3 n_4]} \\ &\quad -3*6\delta^{[n_1 n_2}_{[m_1 m_2}\Gamma_{m_3]}{}^{n_3 n_4]} - 1*4\delta^{[n_1 n_2 n_3}_{m_1 m_2 m_3}\Gamma^{n_4]}.\end{aligned} \tag{A.35}$$

[2] More precisely, the symmetry properties are true for $\Gamma^{(k)}\mathcal{C}$.

A.4 Cohomology and homology classes

A p-form F_p is said to be *closed* iff $\mathrm{d}F_p = 0$. It is *exact* iff there exists a (p-1)-form G_{p-1} such that $F_p = \mathrm{d}G_{p-1}$. Obviously, using the property of the derivation (A.21), an exact form is closed. But the converse needs not be true.

Let's define the group Z^p of closed p-forms, with additive law, and the group B^p of exact p-forms. Then the group $H^p = Z^p/B^p$ contains the classes of closed forms which are equal up to an exact form. The harmonic forms are examples of closed but not exact forms. A classical result is that there is actually a unique harmonic form in each cohomology class. Thus the number of harmonic p-forms is exactly the dimension of H^p. Finally we notice that, when F is harmonic, so is $*F$. This implies Poincaré duality $H^p \sim H^{n-p}$. Let h^p be the dimension of H^p (Betti number). Then the Euler characteristic χ is

$$\chi = \sum_p (-)^p h^p. \tag{A.36}$$

A p-dimensional submanifold γ_p is a *p-cycle* iff it has no boundary $\partial\gamma_p = 0$. Since a boundary has no boundary, ∂ is nilpotent. Thus the boundaries are cycles. But again, the converse needs not be true, there can be cycles which are not boundaries of some submanifold.

Let's define the group Z_p of p-cycles, and the group B_p of p-boundaries. Then the group $H_p = B_p/Z_p$ contains the classes of p-cycles which are equal up to a boundary.

The analogy between homology and cohomology is striking. Indeed, De Rham's theorem states that $H^p \sim H_p$. Since we expand all our forms on harmonic forms, the Betti numbers, the numbers of such forms, are extremely important to us. These numbers are topological invariants, which can be found by looking for independent p-cycles.

A.5 Almost complex, complex and Kähler manifolds

For a detailed review of the notion of (complex) manifolds, see [90,91]. Here we briefly recall the main results. A differentiable manifold is described by a set of patches, that can overlap. On the overlap of two patches, the two sets of coordinates are related by a C_∞ diffeomorphism. A manifold of even dimension 2n is locally diffeomorph to $R^{2n} = C^n$. If moreover, the changes of coordinates are holomorphic, then the manifold is complex. An other way to define the notion of complex manifold is the following. If there is a globally defined 2-tensor ${J_m}^n$ squaring to -1, the manifold is called almost complex, and J is the almost complex structure. This does not mean that J can be used to define complex coordinates globally. To do so, the almost complex structure must be integrable. This is quantified by its "torsion", the Nijenhuis tensor [59]

$${N_{mn}}^k = {J_m}^l(\nabla_l J^k_n - \nabla_n {J_l}^k) - {J_n}^l(\nabla_l J^k_m - \nabla_m {J_l}^k). \tag{A.37}$$

If N vanishes, then the manifold is complex. This means that complex coordinates can be used safely. Let's denote them $z^\alpha, \bar{z}^{\bar{\alpha}}$, $\alpha, \bar{\alpha}$ taking values in $1,2,3$ and $\bar{1},\bar{2},\bar{3}$. The complex structure takes the form

$${J_\alpha}^\beta = +i{\delta_\alpha}^\beta \quad , \quad {J_{\bar{\alpha}}}^{\bar{\beta}} = -i{\delta_{\bar{\alpha}}}^{\bar{\beta}} \quad , \quad {J_\alpha}^{\bar{\beta}} = {J_{\bar{\alpha}}}^\beta = 0. \tag{A.38}$$

It is always possible to choose a hermitian metric for which only the mixed components are non-vanishing. Lowering the indices of the complex structure with this hermitian metric, one can check that $J_{mn} = -J_{nm}$. It is thus natural to define a 2-form out of J

$$J = \frac{1}{2!} J_{mn} \mathrm{d}y^m \mathrm{d}y^n. \tag{A.39}$$

If J is closed, it is called the Kähler class and the manifold is called Kähler. Some very important properties come from this assumption

$$\mathrm{d}J = 0. \tag{A.40}$$

First of all, in complex component, (A.40) can be rewritten

$$\partial_\alpha g_{\beta\bar{\beta}} = \partial_\beta g_{\alpha\bar{\beta}} \tag{A.41}$$

which means that the metric can be written in terms of a (Kähler) potential K

$$g_{\alpha\bar{\alpha}} = \partial_\alpha \partial_{\bar{\alpha}} K. \tag{A.42}$$

This in turns implies that the only non-vanishing Christoffell symbols are completely pure in their indices

$$\Gamma_{\alpha\beta}{}^{\gamma} = g^{\gamma\bar{\gamma}} \partial_\alpha g_{\beta\bar{\gamma}}. \tag{A.43}$$

For the Riemann tensor, it can be checked that the only non-vanishing component is $R_{\alpha\bar{\alpha}\beta\bar{\beta}}$ and the ones obtained by complex conjugation or using the symmetry property

$$R_{[mnp]}{}^{q} = 0. \tag{A.44}$$

A.6 Homogeneous functions of degree 2

Let $F(X)$ be a homogeneous function of degree 2 of the scalars $X^1, X^2, \dots X^n$. This means that F is a polynomial of the X's, with integer powers, such that the sum of all powers is 2. F can be written as

$$F(X) \quad = \quad \sum a_{\lambda_1 \lambda_2 \dots \lambda_n} (X^1)^{\lambda_1} (X^2)^{\lambda_2} \dots (X^n)^{\lambda_n} \tag{A.45}$$

where the sum is other all n-uplets $(\lambda_1 \dots \lambda_n)$ in Z^n such that $\lambda_1 + \dots + \lambda_n = 2$ (with only a finite number of non-zero n-uplets). Obviously such a function has the important property $F(\alpha X^1, \alpha X^2 \dots \alpha X^n) = \alpha^2 F(X^1, X^2 \dots X^n)$, which is precisely the reason why it is used in $N = 2$ supergravities.

Let F_I be the derivative of F with respect to X^I. Then

$$\begin{aligned}
F_1 &= \frac{\partial}{\partial X^1} F = \sum \lambda_1 a_{\lambda_1 \lambda_2 \dots \lambda_n} (X^1)^{\lambda_1 - 1} (X^2)^{\lambda_2} \dots (X^n)^{\lambda_n} \\
&. \\
&. \\
&. \\
F_n &= \frac{\partial}{\partial X^n} F = \sum \lambda_n a_{\lambda_1 \lambda_2 \dots \lambda_n} (X^1)^{\lambda_1} (X^2)^{\lambda_2} \dots (X^n)^{\lambda_n - 1}.
\end{aligned}$$

From this we deduce

$$X^I F_I \quad = \quad \sum (\lambda_1 + \dots + \lambda_n) a_{\lambda_1 \lambda_2 \dots \lambda_n} (X^1)^{\lambda_1} (X^2)^{\lambda_2} \dots (X^n)^{\lambda_n} = 2F. \tag{A.46}$$

Now we successively differentiate (A.46) with respect to X^J, X^K and we obtain

$$X^I F_I \quad = \quad 2F \tag{A.47}$$

$$X^I F_{IJ} \quad = \quad F_J \tag{A.48}$$

$$X^I F_{IJK} \quad = \quad 0. \tag{A.49}$$

Appendix B

Calabi-Yau manifolds

B.1 Main properties of CY_3

We will start from the definition involving a spinor. *A Calabi-Yau manifold admits exactly one covariantly constant spinor.* Using this spinor η, we build the tensor

$$J_m{}^n = -i\eta^\dagger \Gamma_m{}^n \Gamma_7 \eta \tag{B.1}$$

and we want to show that it squares to -1. We evaluate the expression

$$\begin{aligned} J_m{}^n J_n{}^p &= -\eta^{\dagger\, a}\eta^b \eta^{\dagger\, c}\eta^d (\Gamma_m{}^n\Gamma_7)_{ab}(\Gamma_n{}^p\Gamma_7)_{cd} \\ &= -\eta^{\dagger\, a}\eta^b \eta^{\dagger\, c}\eta^d (M_m{}^p)_{abcd}. \end{aligned} \tag{B.2}$$

We rearrange the spinor indices with Fierz method

$$\begin{aligned} (M_m{}^p)_{abcd} &= (k_m{}^p)_{ad}\delta_{cb} + (k_{nm}{}^p)_{ad}(\Gamma^n)_{cb} + (k_{nqm}{}^p)_{ad}(\Gamma^{nq})_{cb} \\ &\quad +(k_{lnqm}{}^p)_{ad}(\Gamma^{lnq})_{cb} + (\tilde{k}_{nqm}{}^p)_{ad}(\Gamma^{nq}\Gamma_7)_{cb} \\ &\quad +(\tilde{k}_{nm}{}^p)_{ad}(\Gamma^n\Gamma_7)_{cb} + (\tilde{k}_m{}^p)_{ad}(\Gamma_7)_{cb}. \end{aligned} \tag{B.3}$$

Considering the symmetry of spinor indices in (B.2), we only compute the coefficients $(k_m{}^p)_{ad}$, $(k_{lnqm}{}^p)_{ad}$ and $(\tilde{k}_{nqm}{}^p)_{ad}$. Multiplying (B.3) respectively by δ^{bc}, $(\Gamma_{lnq})^{bc}$ and $(\Gamma_{nq}\Gamma_7)^{bc}$, we obtain

$$\begin{aligned} k_m{}^p &= \frac{5}{8}\delta^p_m \\ k_{lnqm}{}^p &= -\frac{1}{48}\delta^p_m\Gamma_{lnq} \\ k_{nqm}{}^p &= -\frac{1}{16}\delta^p_m\Gamma_{nq}\Gamma_7 - \frac{1}{4}g_{m[n}\Gamma_{q]}{}^p\Gamma_7 - \frac{1}{4}\delta^p_{[n}\Gamma_{q]m}\Gamma_7. \end{aligned} \tag{B.4}$$

This leads to

$$\frac{3}{2}J_m{}^n J_n{}^p + \frac{1}{16}\delta^p_m J_{nq}J^{nq} = -\frac{9}{8}\delta^p_m \tag{B.5}$$

and the final conclusion

$$J_m{}^n J_n{}^q = -\delta^p_m. \tag{B.6}$$

J is thus an almost complex structure. Since η is covariantly constant (with Christoffell connection), so is J, and the Nijenhuis tensor (A.37) vanishes : the complex structure is integrable and the manifold is complex. Moreover, since J is covariantly constant, it is obviously Kähler. The last defining property, Ricci-flatness, comes from a consistency condition on the spinor. The fact that η is covariantly constant is written, according to (A.11)

$$\nabla_m \eta_\alpha = \partial_m \eta_\alpha - \frac{1}{4}\Gamma_{mab}(\Gamma^{ab})_\alpha{}^\beta \eta_\beta = 0. \tag{B.7}$$

Applying an other covariant derivative and taking the commutator, we find

$$[\nabla_m \, , \, \nabla_n]\eta = -R_{mnpq}\Gamma^{pq}\eta = 0. \tag{B.8}$$

We want a constraint on R, so we multiply successively by Γ^n, Γ^l and $\eta^\dagger$. Using the identities

$$\Gamma^n\Gamma^{pq} = \Gamma^{npq} + 2g^{n[p}\Gamma^{q]} \tag{B.9}$$

$$\Gamma_l\Gamma^{npq} = \Gamma_l{}^{npq} + 3\delta_l^{[n}\Gamma^{pq]} \tag{B.10}$$

$$\Gamma_l\Gamma^q = \Gamma_l{}^q + \delta_l^q \tag{B.11}$$

and the symmetry properties (A.33), we obtain

$$\eta^\dagger\Gamma_l{}^{npq}\eta R_{mnpq} - 2R_{ml} = 0. \tag{B.12}$$

Since R is subject to the Bianchi identity

$$R_{m[npq]} = 0, \tag{B.13}$$

the only surviving term expresses Ricci-flatness

$$R_{mn} = 0. \tag{B.14}$$

For a proof of the structure of the Hodge diamond (2.16), see [91].

B.2 Integrals on CY_3

On the Calabi-Yau manifold one can define complex coordinates ξ^i

$$\xi^1 = \frac{y^1 + iy^2}{\sqrt{2}}; \xi^2 = \frac{y^3 + iy^4}{\sqrt{2}}; \xi^3 = \frac{y^5 + iy^6}{\sqrt{2}}. \tag{B.15}$$

We define the three-dimensional epsilon tensors such that

$$d\xi^\alpha d\xi^\beta d\xi^\gamma d\xi^{\bar\alpha} d\xi^{\bar\beta} d\xi^{\bar\gamma} = \epsilon^{\alpha\beta\gamma}\epsilon^{\bar\alpha\bar\beta\bar\gamma} d^6\xi \tag{B.16}$$

that is to say $\epsilon^{123} = \epsilon^{\bar1\bar2\bar3} = +1$ and $d^6\xi = d\xi^1 d\xi^2 d\xi^3 d\xi^{\bar1} d\xi^{\bar2} d\xi^{\bar3} = -id^6y$. The indices are lowered with the metric as in (A.14), and, keeping in mind that $g_{ij} = g_{\bar i\bar j} = 0$, one has the properties similar to (A.15)

$$\epsilon^{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma} = \sqrt{g}3! \; \ldots \tag{B.17}$$

The relation with the 6-dimensional epsilon tensor is the following. We define the real ϵ-symbol by $\epsilon^{123456} = +1$. The indices are lowered with the metric. It follows that in terms of complex indices one has

$$\epsilon^{\alpha\beta\gamma\bar\alpha\bar\beta\bar\gamma} = -i\epsilon^{\alpha\beta\gamma}\,\epsilon^{\bar\alpha\bar\beta\bar\gamma} \; . \tag{B.18}$$

B.2.1 (1,1)-form sector

Harmonic $(1,1)$-forms are denoted by $(\omega_i)_{\alpha\bar{\beta}}$, i running from 1 to $h^{(1,1)}$. The integrals we abbreviate as

$$\begin{aligned} \mathcal{K} &= \frac{1}{6}\int_Y J\wedge J\wedge J\,, \quad \mathcal{K}_i = \int_Y \omega_i\wedge J\wedge J\,, \\ \mathcal{K}_{ij} &= \int_Y \omega^i\wedge\omega^j\wedge J\,, \quad \mathcal{K}_{ijk} = \int_Y \omega^i\wedge\omega^j\wedge\omega^k\,, \end{aligned} \tag{B.19}$$

where $\mathcal{K}$ is the volume and J is the Kähler form which can be expanded in terms of the basis ω_i as

$$J = v^i\omega_i\ . \tag{B.20}$$

This implies the following identities

$$\begin{aligned} \mathcal{K}_{ijk}v^k = \mathcal{K}_{ij} \qquad &; \qquad \mathcal{K}_{ij}v^j = \mathcal{K}_i \\ \mathcal{K}_i v^i = 6\mathcal{K} & \end{aligned} \tag{B.21}$$

We also define the metric on the complexified Kähler cone

$$g_{ij} = \frac{1}{4\mathcal{K}}\int_Y \omega_i\wedge *\omega_j\ , \tag{B.22}$$

which, using [92]

$$*\omega_i = -J\wedge\omega_i + \frac{\mathcal{K}_i}{4\mathcal{K}}J\wedge J, \tag{B.23}$$

can be rewritten

$$g_{ij} = -\frac{1}{4\mathcal{K}}\left(\mathcal{K}_{ij} - \frac{1}{4\mathcal{K}}\mathcal{K}_i\mathcal{K}_j\right). \tag{B.24}$$

On a Calabi-Yau threefold $H^{2,2}(Y)$ is dual to $H^{(1,1)}(Y)$ and it is useful to introduce the dual basis $\tilde{\omega}^i$ normalized by

$$\int_Y \omega_i\wedge\tilde{\omega}^j = \delta_i^j\ . \tag{B.25}$$

With this normalization the following relations hold

$$g^{ij} = 4\mathcal{K}\int_Y \tilde{\omega}^i\wedge *\tilde{\omega}^j\,, \quad *\omega_i = 4\mathcal{K}g_{ij}\tilde{\omega}^j\,, \quad *\tilde{\omega}^i = \frac{1}{4\mathcal{K}}\,g^{ij}\omega_j\,, \quad \omega_i\wedge\omega_j \sim \mathcal{K}_{ijk}\tilde{\omega}^k\,, \tag{B.26}$$

where the symbol $\sim$ denotes the fact that the quantities are in the same cohomology class.

B.2.2 3-form sector

There are two standard choices of basis for the 3-form sector. One is obviously complex, with $(\eta_a)_{\alpha\beta\bar{\gamma}}$, a running from 1 to $h^{(2,1)}$, a basis for the $(2,1)$-forms, and $\Omega_{\alpha\beta\gamma}$ is the unique holomorphic $(3,0)$-form. The other choice is a complete set of $2(h^{(2,1)}+1)$ real forms α_A, β^A, A running from 0 to $h^{(2,1)}$. This basis is orthonormal in the following sense

$$\int_Y \alpha_A\wedge\alpha_B = \int_Y \beta^A\wedge\beta^B = 0 \tag{B.27}$$

$$\int_Y \alpha_A\wedge\beta^B = -\int_Y \beta^B\wedge\alpha_A = \delta_A^B. \tag{B.28}$$

The $(3,0)$-form Ω can be expanded on this basis with coefficients to be interpreted later

$$\Omega = z^A \alpha_A - \mathcal{F}_A \beta^A. \tag{B.29}$$

Since Ω is covariantly constant, $||\Omega||^2$ defined by

$$||\Omega||^2 \equiv \frac{1}{3!}\Omega_{\alpha\beta\gamma}\bar{\Omega}^{\alpha\beta\gamma} \tag{B.30}$$

is a constant on the Calabi-Yau manifold.

B.3 Lichnerowicz's equation

Consider a deformation of the metric $g_{mn} = g^0_{mn} + \delta g_{mn}$ such that g^0_{mn} is hermitian with vanishing Ricci tensor. Using invariance under diffeomorphisms and tracelessness of the metric, we are allowed to impose the following constraints on δg

$$g^{0\,mn}\delta g_{mn} = 0 \tag{B.31}$$

$$\nabla^m \delta g_{mn} = 0. \tag{B.32}$$

The first order variation of the Christoffel symbols can be written as

$$\delta\Gamma_{mn}{}^p = \frac{1}{2}g^{0\,pl}(\nabla_m\delta g_{nl} + \nabla_n\delta g_{ml} - \nabla_l\delta g_{mn}) \tag{B.33}$$

and leads to the variation of the Riemann tensor

$$\delta R_{mnp}{}^q = \nabla_m\delta\Gamma_{np}{}^q - \nabla_n\delta\Gamma_{mp}{}^q. \tag{B.34}$$

Since we want our manifold to remain Calabi-Yau, we have to impose

$$\delta R_{mn} = 0. \tag{B.35}$$

From (B.31), one can immediately see that

$$\delta\Gamma_{lm}{}^l = 0. \tag{B.36}$$

Plugging this in (B.34) and using (B.32), we find Lichnerowicz equation

$$\nabla^l\nabla_l\delta g_{mn} - [\nabla^l\,,\,\nabla_m]\delta g_{ln} - [\nabla^l\,,\,\nabla_n]\delta g_{lm} = 0. \tag{B.37}$$

This splits into two equations, one on the mixed variations and one on the pure variations. Taking

$$\delta g_{\alpha\bar{\alpha}} = -iv^i(\omega_i)_{\alpha\bar{\alpha}} \tag{B.38}$$

obviously solves (B.37), but for pure variations, the problem is more tricky. First of all there are no (2,0)-forms on the Calabi-Yau, and second of all, $\delta g_{\alpha\beta}$ is symmetric. We show now that

$$\delta g_{\alpha\beta} = \frac{i}{||\Omega||^2}\bar{z}^a(\bar{\eta}_a)_{\alpha\bar{\beta}\bar{\gamma}}\Omega^{\bar{\beta}\bar{\gamma}}{}_\beta \tag{B.39}$$

is indeed symmetric and solution to (B.37). Consider

$$L_{\alpha\beta} \equiv (\bar{\eta}_a)_{\alpha\bar{\beta}\bar{\gamma}}\Omega^{\bar{\beta}\bar{\gamma}}{}_\beta - (\bar{\eta}_a)_{\alpha\bar{\beta}\bar{\gamma}}\Omega^{\bar{\beta}\bar{\gamma}}{}_\beta \tag{B.40}$$

and multiply this equation by $\bar{\Omega}^\beta{}_{\bar{\lambda}\bar{\mu}}$. We find

$$L_{\alpha\beta}\bar{\Omega}^\beta{}_{\bar{\lambda}\bar{\mu}} = -2(\bar{\eta}_a)^{\bar{\rho}}{}_{\bar{\rho}\bar{\lambda}}g_{\alpha\bar{\mu}} - 2(\bar{\eta}_a)^{\bar{\rho}}{}_{\bar{\mu}\bar{\rho}}g_{\alpha\bar{\lambda}}. \tag{B.41}$$

From this we can see that if $(\bar{\eta}_a)^{\bar{\rho}}{}_{\bar{\rho}\bar{\lambda}} = 0$, $L_{\alpha\beta} = 0$. Conversely, by contracting with $g^{\alpha\bar{\lambda}}$ we obtain

$$L_{\alpha\beta}\bar{\Omega}^{\beta\alpha}{}_{\bar{\mu}} = 4(\bar{\eta}_a)^{\bar{\rho}}{}_{\bar{\rho}\bar{\mu}} \tag{B.42}$$

such that if $L_{\alpha\beta} = 0$, then $(\bar{\eta}_a)^{\bar{\rho}}{}_{\bar{\rho}\bar{\mu}} = 0$. Finally, $\delta g_{\alpha\beta}$ is symmetric iff $(\bar{\eta}_a)^{\bar{\rho}}{}_{\bar{\rho}\bar{\mu}} = 0$. Let us write the fact that $\bar{\eta}_a$ is harmonic (A.25)

$$\begin{aligned} \nabla^m\nabla_m(\bar{\eta}_a)_{\alpha\bar{\beta}\bar{\gamma}} \quad &- \quad [\nabla^m\,,\,\nabla_\alpha](\bar{\eta}_a)_{m\bar{\beta}\bar{\gamma}} \\ &- \quad [\nabla^m\,,\,\nabla_{\bar{\beta}}](\bar{\eta}_a)_{\alpha m\bar{\gamma}} \\ &- \quad [\nabla^m\,,\,\nabla_{\bar{\gamma}}](\bar{\eta}_a)_{\alpha\bar{\beta}m} = 0. \end{aligned} \tag{B.43}$$

We want to compute

$$L_{\bar{\gamma}} \equiv \nabla^m\nabla_m(\bar{\eta}_a)^{\bar{\beta}}{}_{\bar{\beta}\bar{\gamma}} \quad - \quad [\nabla^m\,,\,\nabla_{\bar{\gamma}}](\bar{\eta}_a)^{\bar{\beta}}{}_{\bar{\beta}\bar{\gamma}}, \tag{B.44}$$

therefore we contract (B.43) with $g^{\alpha\bar{\beta}}$. To evaluate the commutators, we use (A.5) and the Ricci flatness. We find

$$L_{\bar{\gamma}} = -R^{\beta}{}_{\alpha\bar{\gamma}\delta}(\bar{\eta}_a)_{\beta}{}^{\alpha\delta} - R^{\bar{\delta}}{}_{\bar{\beta}\bar{\gamma}\delta}(\bar{\eta}_a)^{\bar{\beta}}{}_{\bar{\delta}}{}^{\delta} = 0. \tag{B.45}$$

This means that $(\bar{\eta}_a)^{\bar{\beta}}{}_{\bar{\beta}\bar{\gamma}}$ is a harmonic 1-form. Since there are no such forms on a Calabi-Yau manifold, $(\bar{\eta}_a)^{\bar{\beta}}{}_{\bar{\beta}\bar{\gamma}}$ is zero which is the final proof for the symmetry of $\delta g_{\alpha\beta}$.

Lichnerowicz equation on $\delta g_{\alpha\beta}$ reads

$$\nabla^l\nabla_l\delta g_{\alpha\beta} - [\nabla^\gamma\,,\,\nabla_\alpha]\delta g_{\gamma\beta} - [\nabla^\gamma\,,\,\nabla_\beta]\delta g_{\gamma\alpha} = 0. \tag{B.46}$$

Plugging the expression (2.21) for $\delta g_{\alpha\beta}$, and contracting with $\bar{\Omega}_{\bar{\beta}\bar{\gamma}}{}^{\beta}$, this is equivalent to

$$\nabla^m\nabla_m(\bar{\eta}_a)_{\alpha\bar{\beta}\bar{\gamma}} \quad + \quad 2R^{\gamma}{}_{\alpha}{}^{\bar{\rho}}{}_{\bar{\beta}}(\bar{\eta}_a)_{\gamma\bar{\rho}\bar{\gamma}} - 2R^{\gamma}{}_{\alpha}{}^{\bar{\rho}}{}_{\bar{\gamma}}(\bar{\eta}_a)_{\gamma\bar{\rho}\bar{\beta}} = 0. \tag{B.47}$$

This is exactly the equation of harmonicity of $\bar{\eta}_a$ (B.43). Thus (2.21) is indeed solution to Lichnerowicz equation.

B.4 Compactification of the Ricci scalar

We perform an expansion of the Ricci scalar up to order 2 in the moduli. The components of the metric and its inverse are

$$g_{\alpha\beta} = 0 + \bar{z}^a(\bar{b}_a)_{\alpha\beta} \tag{B.48}$$

$$g_{\alpha\bar{\alpha}} = \overset{0}{g}_{\alpha\bar{\alpha}} - iv^i(\omega_i)_{\alpha\bar{\alpha}} \tag{B.49}$$

$$g^{\alpha\beta} = 0 - z^a(b_a)_{\bar{\alpha}\bar{\beta}}g^{0\,\alpha\bar{\alpha}}g^{0\,\beta\bar{\beta}} \tag{B.50}$$

$$g^{\alpha\bar{\alpha}} = g^{0\,\alpha\bar{\alpha}} + iv^i(\omega_i)_{\beta\bar{\beta}}g^{0\,\alpha\bar{\beta}}g^{0\,\beta\bar{\alpha}} \tag{B.51}$$

with

$$(\bar{b}_a)_{\alpha\beta} = \frac{i}{||\Omega||^2}(\bar{\eta}_a)_{\alpha\bar{\beta}\bar{\gamma}}\Omega^{\bar{\beta}\bar{\gamma}}{}_{\beta}. \tag{B.52}$$

Apart from the purely space-time ones, the non-vanishing Christoffell symbols are

$$\begin{aligned}
\Gamma_{\mu\alpha}{}^{\beta} &= \frac{1}{2}(\omega_i)_{\gamma\bar{\gamma}}(\omega_j)_{\alpha\bar{\beta}}g^{0\,\gamma\bar{\beta}}g^{0\,\beta\bar{\gamma}}v^i\partial_\mu v^j - \frac{i}{2}(\omega_i)_{\alpha\bar{\beta}}g^{0\,\beta\bar{\beta}}\partial_\mu v^i \\
&\quad -\frac{1}{2}(b_a)_{\bar{\beta}\bar{\gamma}}(\bar{b}_b)_{\alpha\gamma}g^{0\,\gamma\bar{\beta}}g^{0\,\beta\bar{\gamma}}z^a\partial_\mu\bar{z}^b && \text{(B.53)} \\
\Gamma_{\mu\alpha}{}^{\bar{\beta}} &= \frac{1}{2}(\bar{b}_a)_{\alpha\beta}g^{0\,\beta\bar{\beta}}\partial_\mu\bar{z}^a + \frac{i}{2}(\omega_i)_{\gamma\bar{\gamma}}(\bar{b}_a)_{\alpha\beta}g^{0\,\beta\bar{\gamma}}g^{0\,\gamma\bar{\beta}}v^i\partial_\mu\bar{z}^a \\
&\quad +\frac{i}{2}(\omega_i)_{\alpha\bar{\gamma}}(\bar{b}_a)_{\gamma\beta}g^{0\,\beta\bar{\gamma}}g^{0\,\gamma\bar{\beta}}\partial_\mu v^i\bar{z}^a && \text{(B.54)} \\
\Gamma_{\alpha\beta}{}^{\mu} &= -\frac{1}{2}(\bar{b}_a)_{\alpha\beta}\partial^\mu\bar{z}^a && \text{(B.55)} \\
\Gamma_{\alpha\bar{\beta}}{}^{\mu} &= +\frac{i}{2}(\omega_i)_{\alpha\bar{\beta}}\partial^\mu v^i. && \text{(B.56)}
\end{aligned}$$

The 10-dimensional Ricci scalar has the following decomposition

$$\begin{aligned}
R_{10} &= R_4 + g^{\mu\nu}R_{\mu\alpha\nu}{}^{\alpha} + g^{\alpha\beta}\left(R_{\alpha\mu\beta}{}^{\mu} + R_{\alpha\gamma\beta}{}^{\gamma} + R_{\alpha\bar{\gamma}\beta}{}^{\bar{\gamma}}\right) && \text{(B.57)} \\
&\quad +g^{\alpha\bar{\beta}}\left(R_{\alpha\mu\bar{\beta}}{}^{\mu} + R_{\alpha\gamma\bar{\beta}}{}^{\gamma} + R_{\alpha\bar{\gamma}\bar{\beta}}{}^{\bar{\gamma}}\right) + \text{c.c.} && \text{(B.58)}
\end{aligned}$$

where c.c. means complex conjugate of all terms except R_4. Before going further in this calculation, let's look at a trick that we will use several times. There will appear terms like

$$\sqrt{-g_4}\sqrt{g_6}\nabla_\mu V^\mu. \tag{B.59}$$

Recalling (A.3), we see that this is not exactly a total derivative, because the summation is not on all indices, but only on the space-time ones. A generic expression like (B.59) will be transformed into

$$\sqrt{-g_4}\sqrt{g_6}\nabla_\mu V^\mu \sim -\sqrt{-g_4}V^\mu\partial_\mu\sqrt{g_6}. \tag{B.60}$$

where $\sim$ means equal up to a total space-time derivative. For the derivative of the determinant of the metric, we use (A.4) and we find

$$\partial_\mu\sqrt{g_6} = \frac{1}{2}\sqrt{g_6}\left(g^{\alpha\beta}\partial_\mu g_{\alpha\beta} + g^{\bar{\alpha}\bar{\beta}}\partial_\mu g_{\bar{\alpha}\bar{\beta}} + 2g^{\alpha\bar{\beta}}\partial_\mu g_{\alpha\bar{\beta}}\right), \tag{B.61}$$

which leads to the following expressions for the Ricci scalar

$$\begin{aligned} g^{\mu\nu} R_{\mu\alpha\nu}{}^{\alpha} &\sim \frac{1}{2}\left((\omega_i g)(\omega_j g) - \frac{1}{2}\omega_i\omega_j\right)\partial_\mu v^i \partial^\mu v^j \\ &\quad + \frac{1}{4} b_a \bar{b}_b \partial_\mu z^a \partial^\mu \bar{z}^b \end{aligned} \tag{B.62}$$

$$g^{\alpha\beta} R_{\alpha\mu\beta}{}^{\mu} \sim \frac{1}{2} b_a \bar{b}_b \partial_\mu z^a \partial^\mu \bar{z}^b \tag{B.63}$$

$$g^{\alpha\beta}\left(R_{\alpha\gamma\beta}{}^{\gamma} + R_{\alpha\bar{\gamma}\beta}{}^{\bar{\gamma}}\right) = O(3) \tag{B.64}$$

$$\begin{aligned} g^{\alpha\bar{\beta}} R_{\alpha\mu\bar{\beta}}{}^{\mu} &\sim \frac{1}{2}\left((\omega_i g)(\omega_j g) - \frac{1}{2}\omega_i\omega_j\right)\partial_\mu v^i \partial^\mu v^j \\ &\quad - \frac{1}{4} b_a \bar{b}_b \partial_\mu z^a \partial^\mu \bar{z}^b \end{aligned} \tag{B.65}$$

$$g^{\alpha\bar{\beta}} R_{\alpha\gamma\bar{\beta}}{}^{\gamma} = -\frac{1}{4}\left((\omega_i g)(\omega_j g) - \omega_i\omega_j\right)\partial_\mu v^i \partial^\mu v^j \tag{B.66}$$

$$\begin{aligned} g^{\alpha\bar{\beta}} R_{\alpha\bar{\gamma}\bar{\beta}}{}^{\bar{\gamma}} &= -\frac{1}{4}(\omega_i g)(\omega_j g)\partial_\mu v^i \partial^\mu v^j \\ &\quad - \frac{1}{4} b_a \bar{b}_b \partial_\mu z^a \partial^\mu \bar{z}^b \end{aligned} \tag{B.67}$$

where $O(3)$ means of order in the moduli greater or equal to 3. Finally we obtain

$$\begin{aligned} \int \mathrm{d}^{10}\hat{x}\sqrt{-\hat{g}}\hat{R} = \int \mathrm{d}^4 x\sqrt{-g_4} & \left(\mathcal{K} R_4 + P_{ij}\partial_\mu v^i \partial^\mu v^j \right. \\ & \left. + Q_{ab}\partial_\mu z^a \partial^\mu \bar{z}^b\right), \end{aligned} \tag{B.68}$$

with

$$P_{ij} = \int_Y (\omega_i g)(\omega_j g) - \frac{1}{2}\omega_i\omega_j \tag{B.69}$$

$$Q_{ab} = \frac{1}{2}\int_Y b_a \bar{b}_b \tag{B.70}$$

$$(\omega_i g) = (\omega_i)_{\alpha\bar{\beta}} g^{0\,\alpha\bar{\beta}} \tag{B.71}$$

$$\omega_i\omega_j = (\omega_i)_{\alpha\bar{\alpha}}(\omega_j)_{\beta\bar{\beta}} g^{0\,\alpha\bar{\beta}} g^{0\,\beta\bar{\alpha}} \tag{B.72}$$

$$b_a\bar{b}_b = (b_a)_{\bar{\alpha}\bar{\beta}}(\bar{b}_b)_{\alpha\beta} g^{0\,\alpha\bar{\beta}} g^{0\,\beta\bar{\alpha}}. \tag{B.73}$$

Observe that in components, we have

$$\mathcal{K}_{ij} = \int_Y \omega_i\omega_j - (\omega_i g)(\omega_j g) \tag{B.74}$$

which leads to

$$P_{ij} = -\mathcal{K}_{ij} + \frac{1}{2}\int_Y \omega_i\omega_j. \tag{B.75}$$

Writing (B.22) in components, we find

$$g_{ij} = -\frac{1}{4\mathcal{K}}\int_Y \omega_i \omega_j \tag{B.76}$$

$$P_{ij} = -\mathcal{K}_{ij} - 2\mathcal{K} g_{ij} = +2\mathcal{K} g_{ij} - \frac{1}{4\mathcal{K}}\mathcal{K}_i\mathcal{K}_j. \tag{B.77}$$

where we have used (B.24) for the last equation.

B.5 Moduli space

For a detailed account of the moduli space of Calabi-Yau threefold, see [93]. One of the main results is that the moduli space $\mathcal{M}$ splits into the product of two spaces : $\mathcal{M}_{1,1}$ corresponds to the deformations of the Kähler class , which are parameterized by the harmonic $(1,1)$-forms, and $\mathcal{M}_{2,1}$ corresponds to the deformations of the complex structure, parameterized by the harmonic $(2,1)$-forms. Both spaces are special Kähler [94]. The whole moduli space is thus

$$\mathcal{M} = \mathcal{M}_{1,1} \times \mathcal{M}_{2,1}. \tag{B.78}$$

B.5.1 Kähler class moduli space

$\mathcal{M}_{1,1}$ describes the sector of the scalars of the vector multiplets for type IIA, and the sector of the scalars of hypermultiplets for type IIB. Recall that the NS-NS 2-form B_2 is present in type IIA and type IIB, and that it combines with the Kähler class in the following way (2.56)

$$B_2 + iJ \quad \longrightarrow \quad t^i = b^i + iv^i \tag{B.79}$$

The metric for this sector is

$$g_{ij} = \frac{1}{4\mathcal{K}}\int \omega_i \wedge *\omega_j = -\frac{1}{4\mathcal{K}}\left(\mathcal{K}_{ij} - \frac{1}{4\mathcal{K}}\mathcal{K}_i\mathcal{K}_j\right); \tag{B.80}$$

from this we deduce that it is Kähler with Kähler potential K

$$e^{-K} = 8\mathcal{K} \ . \tag{B.81}$$

This potential can be written in terms of a prepotential $\mathcal{F}$

$$e^{-K} = i\left(\bar{X}^I\mathcal{F}_I - X^I\bar{\mathcal{F}}_I\right), \qquad \mathcal{F}_I \equiv \frac{\partial}{\partial X^I}\mathcal{F} \tag{B.82}$$

with

$$\mathcal{F} = -\frac{1}{3!}\frac{\mathcal{K}_{ijk}X^iX^jX^k}{X^0}. \tag{B.83}$$

Here the index I is running from 0 to $h^{1,1}$, and we have defined the X^I in terms of the special coordinates t^i as $X^I = (1, t^i)$. The potential (B.82) has a symplectic invariance and the corresponding manifold is called Special Kähler.

The vectors of the vector multiplets in type IIA couple through a matrix $\mathcal{N}$ given below. This matrix is also the one appearing in the hypermultiplet sector of type IIB. It is defined by

$$\mathcal{N}_{IJ} = \bar{\mathcal{F}}_{IJ} + \frac{2i}{X^P \mathrm{Im}\,\mathcal{F}_{PQ}X^Q}\mathrm{Im}\,\mathcal{F}_{IK}X^K\mathrm{Im}\,\mathcal{F}_{JL}X^L \tag{B.84}$$

and its real and imaginary parts are

$$
\begin{aligned}
\mathrm{Re}\,\mathcal{N}_{00} &= -\frac{1}{3}\mathcal{K}_{ijk}b^ib^jb^k\,, &\quad \mathrm{Im}\,\mathcal{N}_{00} &= -\mathcal{K} + \left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right)b^ib^j\ ,\\
\mathrm{Re}\,\mathcal{N}_{i0} &= \frac{1}{2}\mathcal{K}_{ijk}b^jb^k\ , &\quad \mathrm{Im}\,\mathcal{N}_{i0} &= -\left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right)b^j\ ,\\
\mathrm{Re}\,\mathcal{N}_{ij} &= -\mathcal{K}_{ijk}b^k\ , &\quad \mathrm{Im}\,\mathcal{N}_{ij} &= \left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right)\ ,
\end{aligned}
\tag{B.85}
$$

We will also need the inverse of the imaginary part of $\mathcal{N}$

$$
(\mathrm{Im}\,\mathcal{N})^{-1} = -\frac{1}{\mathcal{K}}\begin{pmatrix} 1 & b^i \\ b^i & \frac{g^{ij}}{4} + b^ib^j \end{pmatrix}
\tag{B.86}
$$

and the inverse metric has the explicit form

$$
g^{ij} = -4\mathcal{K}\left(\mathcal{K}^{ij} - \frac{v^iv^j}{2\mathcal{K}}\right)
\tag{B.87}
$$

in terms of $\mathcal{K}^{ij}$ defined by

$$
\mathcal{K}^{ij}\mathcal{K}_{jk} = \delta^i_k.
\tag{B.88}
$$

Multiplying (B.88) by v^k, we also obtain the following useful relation

$$
\mathcal{K}^{ij}\mathcal{K}_j = v^i.
\tag{B.89}
$$

B.5.2 Complex structure moduli space

According to Kodaira's formula [95], $\frac{\partial}{\partial z^a}\Omega$ is in $H^{3,0} + H^{2,1}$

$$
\frac{\partial}{\partial z^a}\Omega = k_a\Omega + i\eta_a.
\tag{B.90}
$$

where η_a is the basis for $(2,1)$-forms used in (B.52). The metric for the scalars z^a was found in (2.35), and can be rewritten

$$
g_{a\bar{b}} = -\frac{i}{\mathcal{K}||\Omega||^2}\int_Y \eta_a \wedge \bar{\eta}_b.
\tag{B.91}
$$

From this and (B.90) we deduce that the metric for complex structure deformations g_{ab} is also Kähler with Kähler potential given by

$$
e^{-K} \ = \ i\int_Y \Omega\wedge\bar{\Omega} \ = \ \mathcal{K}\,||\Omega||^2.
\tag{B.92}
$$

It is argued that Ω can be taken homogeneous of degree one [93], so the coordinates z^A in the expansion (B.29) are actually projective. Again this potential can be written in terms of a prepotential $\mathcal{F}$ as

$$
e^{-K} = i\left(\bar{z}^A\mathcal{F}_A - z^A\bar{\mathcal{F}}_A\right), \qquad \mathcal{F}_A \equiv \frac{\partial}{\partial z^A}\mathcal{F}
\tag{B.93}
$$

where $\mathcal{F}_A$, the one appearing in (B.29), is a function of the z^A. The moduli space for the Kähler class deformations is thus Special Kähler.

In order to evaluate the integrals in the reduction we need to recall that the Hodge-dual basis $(*\alpha_A, *\beta^A)$ is related to (α_A, β^A) via

$$
*\alpha_A = A_A{}^B\,\alpha_B + B_{AB}\,\beta^B\ , \qquad *\beta^A = C^{AB}\,\alpha_B + D^A{}_B\,\beta^B\ ,
\tag{B.94}
$$

where $A,\ B,\ C,\ D$ are some unknown matrices. The relation

$$\int \alpha_A \wedge *\beta^B = \int \beta^B \wedge *\alpha_A \tag{B.95}$$

implies

$$A_A{}^B = -D^B{}_A, \tag{B.96}$$

and similarly it can be obtained that B and C are symmetric. Following [96–99], we will show that these matrices can be expressed in terms of the moduli. To this end, we start by noticing the identities

$$* \Omega \quad = \quad -i\Omega \tag{B.97}$$

$$*\pi \quad = \quad +i\pi \tag{B.98}$$

for the $(3,0)$-form Ω and any $(2,1)$-form π. (B.97) can be expressed directly using the expansion (B.29) and the definitions (B.94)

$$z^A A_A{}^B - \mathcal{F}_A C^{AB} \quad = \quad -iz^B \tag{B.99}$$

$$z^A B_{AB} + \mathcal{F}_A A_B{}^A \quad = \quad i\mathcal{F}_B. \tag{B.100}$$

Since the forms η_a are $(2,1)$, they do not contribute to the integral

$$\int_Y \partial_a \Omega \wedge \bar{\Omega} = k_a \int_Y \Omega \wedge \bar{\Omega} \tag{B.101}$$

which enables us to evaluate k_a

$$k_a = \frac{1}{< z|\bar{z} >} \mathrm{Im}\, \mathcal{F}_{aB} \bar{z}^B = -\partial_a K \tag{B.102}$$

where we defined the inner product

$$< F, \bar{G} >= \mathrm{Im}\, \mathcal{F}_{AB} F^A \bar{G}^B \tag{B.103}$$

and we have used formula (A.48) for homogeneous functions. Adapting the argument in [93] to the coordinate z^0, we know that $\frac{\partial}{\partial z^0}\Omega \in H^{3,0} + H^{2,1}$, and we define the $(2,1)$ piece by

$$\frac{\partial}{\partial z^0}\Omega = \frac{1}{< z|\bar{z} >} \mathrm{Im}\, \mathcal{F}_{0B} \bar{z}^B \Omega + i\eta_0. \tag{B.104}$$

With the general expression

$$\frac{\partial}{\partial z^A}\Omega = \alpha_A - \mathcal{F}_{AB}\beta^B \tag{B.105}$$

one can expand η_a on the $(\alpha_A\,,\, \beta^B)$ basis, and imposing (B.98) for η_A, we find the equations

$$A_A{}^B - \mathcal{F}_{AC} C^{CB} \quad = \quad i\delta_A^B - \frac{2i}{< z|\bar{z} >} \mathrm{Im}\, \mathcal{F}_{AC} \bar{z}^C z^B \tag{B.106}$$

$$B_{AB} + \mathcal{F}_{AC} A_B{}^C \quad = \quad -i\mathcal{F}_{AB} + \frac{2i}{< z|\bar{z} >} \mathrm{Im}\, \mathcal{F}_{AC} \bar{z}^C \mathcal{F}_B. \tag{B.107}$$

Remark that multiplying (B.106) and (B.107) by z^A one recovers (B.99) and (B.100). Separating the real and imaginary parts of (B.106) and (B.107) we find the expressions of the matrices

$$A_A{}^B = -\mathrm{Re}\,\mathcal{F}_{AC}\left(\mathrm{Im}\,\mathcal{F}^{-1}\right)^{CB} + \frac{\bar{z}^B\mathcal{F}_A + z^B\bar{\mathcal{F}}_A}{< z|\bar{z} >} \tag{B.108}$$

$$B_{AB} = \mathrm{Im}\,\mathcal{F}_{AB} + \mathrm{Re}\,\mathcal{F}_{AC}\left(\mathrm{Im}\,\mathcal{F}^{-1}\right)^{CD}\mathrm{Re}\,\mathcal{F}_{DB} - \frac{\bar{\mathcal{F}}_A\mathcal{F}_B + \mathcal{F}_A\bar{\mathcal{F}}_B}{< z|\bar{z} >} \tag{B.109}$$

$$C^{AB} = -\left(\mathrm{Im}\,\mathcal{F}^{-1}\right)^{AB} + \frac{\bar{z}^A z^B + z^A\bar{z}^B}{< z|\bar{z} >}. \tag{B.110}$$

We introduce the matrix $\mathcal{M}$

$$\mathcal{M}_{AB} = \bar{\mathcal{F}}_{AB} + \frac{2i}{< z|z >}\mathrm{Im}\,\mathcal{F}_{AC}z^C\mathrm{Im}\,\mathcal{F}_{BD}z^D \tag{B.111}$$

and we give the expression of A, B, C in terms of $\mathcal{M}$

$$\begin{aligned} A &= (\mathrm{Re}\,\mathcal{M})\,(\mathrm{Im}\,\mathcal{M})^{-1} \ , \\ B &= -\,(\mathrm{Im}\,\mathcal{M}) - (\mathrm{Re}\,\mathcal{M})\,(\mathrm{Im}\,\mathcal{M})^{-1}\,(\mathrm{Re}\,\mathcal{M}) \ , \\ C &= (\mathrm{Im}\,\mathcal{M})^{-1} \ . \end{aligned} \tag{B.112}$$

Appendix C

Type II supergravities on CY_3 with NS-form fluxes

C.1 Type IIA with NS fluxes

In this section we briefly recall the results of [7] for the compactification of type IIA supergravity on Calabi-Yau three-folds Y when background NS fluxes are turned on.

The bosonic spectrum of type IIA supergravity in ten dimensions features the following fields: the graviton $\hat{g}_{MN}$, a two-form $\hat{B}_2$ and the dilaton $\hat{\phi}$ in the NS-NS sector and a one form $\hat{A}_1$ and a three-form $\hat{C}_3$ in the RR sector. The action governing the interactions of these fields can be written as [3]

$$
\begin{aligned}
S &= \int e^{-2\hat{\phi}} \left(-\frac{1}{2}\hat{R} *\mathbf{1} + 2d\hat{\phi}\wedge *d\hat{\phi} - \frac{1}{4}\hat{H}_3 \wedge *\hat{H}_3 \right) \\
&\quad -\frac{1}{2}\int \left(\hat{F}_2 \wedge *\hat{F}_2 + \hat{F}_4 \wedge *\hat{F}_4 \right) + \frac{1}{2}\int \hat{H}_3 \wedge \hat{C}_3 \wedge d\hat{C}_3 \, ,
\end{aligned} \tag{C.1}
$$

where

$$
\hat{H}_3 = d\hat{B}_2 \, , \qquad \hat{F}_2 = d\hat{A}_1 \, , \qquad \hat{F}_4 = d\hat{C}_3 - \hat{A}_1 \wedge \hat{H}_3 . \tag{C.2}
$$

Upon compactification on a Calabi-Yau three-fold the four-dimensional spectrum can be read from the expansion of the ten-dimensional fields in the Calabi-Yau harmonic forms

$$
\begin{aligned}
\hat{A}_1 &= A^0 \, , \\
\hat{C}_3 &= C_3 + A^i \wedge \omega_i + \xi^A \alpha_A + \tilde{\xi}_A \beta^A \, , \\
\hat{B}_2 &= B_2 + b^i \omega_i \, .
\end{aligned} \tag{C.3}
$$

Correspondingly, in $D = 4$ we find a three-form C_3, a two-form B_2, the vector fields (A^0, A^i) and the scalars $b^i, \xi^A, \tilde{\xi}_A$. Together with the Kähler class and complex structure deformations v^i and z^a these fields combine into a gravity multiplet $(G_{\mu\nu}, A^0)$, $h^{(1,1)}$ vector multiplets (A^i, v^i, b^i), $i = 1, \ldots, h^{(1,1)}$, $h^{(1,2)}$ hyper-multiplets $(z^a, \xi^a, \tilde{\xi}_a)$, $a = 1, \ldots, h^{(1,2)}$ and a tensor multiplet $(B_2, \phi, \xi^0, \tilde{\xi}_0)$.

We assume that turning on background fluxes does not change the light spectrum and thus the only modification in the KK Ansatz is a shift in the field strength of $\hat{B}_2$

$$
\hat{H}_3 = H_3 + db^i \wedge \omega_i + p^A \alpha_A + q_A \beta^A \, . \tag{C.4}
$$

This leads to the following expressions for the different terms appearing in the ten-dimensional

action (2.36)

$$
\begin{aligned}
-\frac{1}{4}\int_Y \hat{H}_3 \wedge *\hat{H}_3 &= -\frac{\mathcal{K}}{4} H_3 \wedge *H_3 - \mathcal{K} g_{ij} db^i \wedge *db^j - V * \mathbf{1} \ , \\
-\frac{1}{2}\int_Y \hat{F}_2 \wedge *\hat{F}_2 &= -\frac{\mathcal{K}}{2} dA^0 \wedge *dA^0 \ , \\
-\frac{1}{2}\int_Y \hat{F}_4 \wedge *\hat{F}_4 &= -\frac{\mathcal{K}}{2} (dC_3 - A^0 \wedge H_3) \wedge *(dC_3 - A^0 \wedge H_3) \qquad \text{(C.5)} \\
& -2\mathcal{K} g_{ij} (dA^i - A^0 db^i) \wedge *(dA^j - A^0 db^j) \\
& +\frac{1}{2} \left(\mathrm{Im}\,\mathcal{M}^{-1}\right)^{AB} \left[D\tilde{\xi}_A + \mathcal{M}_{AC} D\xi^C\right] \wedge * \left[D\tilde{\xi}_B + \bar{\mathcal{M}}_{BD} D\xi^D\right] \ , \\
\frac{1}{2}\int_Y \hat{H}_3 \wedge \hat{C}_3 \wedge d\hat{C}_3 &= -\frac{1}{2} H_3 \wedge (\xi^A d\tilde{\xi}_A - \tilde{\xi}_A d\xi^A) + \frac{1}{2} db^i \wedge A^j \wedge dA^k \mathcal{K}_{ijk} \\
& + dC_3 \wedge \left(p^A \tilde{\xi}_A - q_A \xi^A\right) \ .
\end{aligned}
$$

Even from this stage one can notice that some of the fields effectively became charged

$$
D\xi^A = d\xi^A - p^A A^0 \quad , \quad D\tilde{\xi}_A = d\tilde{\xi}_A - q_A A^0 \ , \qquad \text{(C.6)}
$$

and a potential term is induced

$$
V = -\frac{1}{4} \mathrm{e}^{-\hat{\phi}} \left(q + \mathcal{M} p\right) \mathrm{Im}\,\mathcal{M}^{-1} \left(q + \bar{\mathcal{M}} p\right) \ . \qquad \text{(C.7)}
$$

1

Next, the compactification proceeds as usually by dualizing the fields C_3 and B_2 to a constant and to a scalar respectively. We do not perform these steps here, but we just recall the final results. (for more details see [7, 64]). First the dualization of C_3 to a constant e results in

$$
\mathcal{L}_e = \mathcal{L}_{C_3} = -\frac{\mathrm{e}^{4\phi}}{2\mathcal{K}} \left(p^A \tilde{\xi}_A - q_A \xi^A + e\right)^2 * \mathbf{1} + \left(p^A \tilde{\xi}_A - q_A \xi^A + e\right) A^0 \wedge H_3 \ . \qquad \text{(C.8)}
$$

It was shown in [7] that the constant e plays a special role in the case of RR fluxes. however, it is irrelevant for the analysis in this paper and thus we will set it to zero. Dualizing now the two-form B_2, one obtains an axion, which due to the Green-Schwarz term in (C.8) becomes charged and its covariant derivative reads

$$
Da = da - \left(p^A \tilde{\xi}_A - q_A \xi^A\right) A^0 . \qquad \text{(C.9)}
$$

Collecting all terms one can write the final form of the action[2]

$$
\begin{aligned}
S_{IIA} &= \int \Big[-\frac{1}{2} R^* \mathbf{1} - g_{ij} dt^i \wedge *d\bar{t}^j - h_{uv} Dq^u \wedge *Dq^v - V_{IIA} * \mathbf{1} \\
& + \frac{1}{2} \mathrm{Im} \mathcal{N}_{IJ} F^I \wedge *F^J + \frac{1}{2} \mathrm{Re} \mathcal{N}_{IJ} F^I \wedge F^J \Big] \ , \qquad \text{(C.10)}
\end{aligned}
$$

where the potential can be read from (C.7) and (C.8)

$$
V_{IIA} = -\frac{1}{4\mathcal{K}} \mathrm{e}^{2\phi} \left(q + \mathcal{M} p\right) \mathrm{Im}\,\mathcal{M}^{-1} \left(q + \bar{\mathcal{M}} p\right) + \frac{1}{2\mathcal{K}} \mathrm{e}^{4\phi} \left(p^A \tilde{\xi}_A - q_A \xi^A\right)^2 \ , \qquad \text{(C.11)}
$$

[1] For a systematic study of the Calabi-Yau moduli space we refer the reader to the literature [15, 93].

[2] We have further redefined the gauge fields as $A^i \longrightarrow A^i - b^i A^0$ and also appropriately rescaled the metric in order to go to the Einstein frame.

while the metric for the hyper-scalars h_{uv} has the standard form of [22]

$$\begin{aligned} h_{uv} Dq^u \wedge *Dq^v &= d\phi \wedge *d\phi + g_{ab} dz^a \wedge *d\bar{z}^b \\ &+ \frac{e^{4\phi}}{4} \Big[Da + (\tilde{\xi}_A D\xi^A - \xi^A D\tilde{\xi}_A) \Big] \wedge * \Big[Da + (\tilde{\xi}_A D\xi^A - \xi^A D\tilde{\xi}_A) \Big] \\ &- \frac{e^{2\phi}}{2} \left(\mathrm{Im}\, \mathcal{M}^{-1} \right)^{AB} \Big[D\tilde{\xi}_A + \mathcal{M}_{AC} D\xi^C \Big] \wedge * \Big[D\tilde{\xi}_B + \bar{\mathcal{M}}_{BD} D\xi^D \Big] \ . \end{aligned} \tag{C.12}$$

C.2 Type IIB with NS fluxes

In this section we recall the compactification of type IIB supergravity on a Calabi-Yau 3-folds with NS 2-form fluxes. The only differences with section 2.3 appear in the expansions of the field strengths which have to take into account H_3 fluxes

$$\hat{H}_3 = H_3 + db^i \wedge \omega_i + \tilde{m}^A \alpha_A - \tilde{e}_A \beta^A . \tag{C.13}$$

Consequently, $\hat{F}_5$ is also modified, according to

$$\begin{aligned} \hat{F}_5 &= \tilde{F}^A \wedge \alpha_A - \tilde{G}_A \beta^A + \left(dD_2^i - db^i \wedge C_2 - c^i H_3 \right) \wedge \omega_i \\ &+ d\rho_i \wedge \tilde{\omega}^i - c^i db^j \wedge \omega_i \wedge \omega_j \end{aligned} \tag{C.14}$$

with

$$\tilde{F}^A = F^A - \tilde{m}^A C_2 \quad ; \quad \tilde{G}_A = G_A - \tilde{e}_A C_2 . \tag{C.15}$$

The straightforward expansion reads

$$\begin{aligned} -\frac{1}{4} \mathrm{e}^{-\hat{\phi}} \int_Y \hat{H}_3 \wedge * \hat{H}_3 &= -\frac{\mathcal{K}}{4} \mathrm{e}^{-\hat{\phi}}\, H_3 \wedge * H_3 - \mathcal{K} \mathrm{e}^{-\hat{\phi}} g_{ij} db^i \wedge * db^j \\ &+ \frac{1}{4} \mathrm{e}^{-\hat{\phi}} \left(\tilde{e} - \mathcal{M} \tilde{m} \right) Im \mathcal{M}^{-1} \left(\tilde{e} - \bar{\mathcal{M}} \tilde{m} \right) * \mathbf{1} \end{aligned} \tag{C.16}$$

$$-\frac{1}{2} \mathrm{e}^{2\hat{\phi}} \int_Y d\hat{l} \wedge * d\hat{l} = -\frac{\mathcal{K}}{2} \mathrm{e}^{2\hat{\phi}}\, dl \wedge * dl \ , \tag{C.17}$$

$$\begin{aligned} -\frac{1}{2} \mathrm{e}^{\hat{\phi}} \int_Y \hat{F}_3 \wedge * \hat{F}_3 &= -\frac{\mathcal{K}}{2} \mathrm{e}^{\hat{\phi}} \left(dC_2 - l H_3 \right) \wedge * \left(dC_2 - l H_3 \right) \\ &- 2\mathcal{K} \mathrm{e}^{\hat{\phi}} g_{ij} (dc^i - l db^i) \wedge * (dc^j - l db^j) \\ &+ \frac{1}{2} \mathrm{e}^{\hat{\phi}} l^2 \left(\tilde{e} - \mathcal{M} \tilde{m} \right) Im \mathcal{M}^{-1} \left(\tilde{e} - \bar{\mathcal{M}} \tilde{m} \right) * \mathbf{1} \ , \end{aligned} \tag{C.18}$$

$$\begin{aligned} -\frac{1}{4} \int \hat{F}_5 \wedge * \hat{F}_5 &= + \frac{1}{4} Im \mathcal{M}^{-1} \left(\tilde{G} - \mathcal{M} \tilde{F} \right) \wedge * \left(\tilde{G} - \bar{\mathcal{M}} \tilde{F} \right) \\ &- \mathcal{K} g_{ij} d\tilde{D}_2^i \wedge * d\tilde{D}_2^j - \frac{1}{16\mathcal{K}} g^{ij} d\tilde{\rho}_i \wedge * d\tilde{\rho}_j \end{aligned} \tag{C.19}$$

$$\begin{aligned} -\frac{1}{2} \int \hat{A}_4 \wedge \hat{H}_3 \wedge d\hat{C}_2 &= -\frac{1}{2} \mathcal{K}_{ijk} D_2^i \wedge db^j \wedge dc^k - \frac{1}{2} \rho_i \left(dB_2 \wedge dc^i + db^i \wedge dC_2 \right) \\ &+ \frac{1}{2} C_2 \wedge \left(F^A \tilde{e}_A - G_A \tilde{m}^A \right) \end{aligned} \tag{C.20}$$

where $d\tilde{D}_2^i$ and $d\tilde{\rho}_i$ have been defined in (2.70) and (2.71). Since the sector of D_2^i and ρ_i is not modified by the fluxes, the elimination of D_2^i is made in exactly the same way as in section 2.3. However, the sector of C_2 gets modified. We still add the duality Lagrangian

$$+\frac{1}{2}F^A \wedge G_A \tag{C.21}$$

which imposes the new self-duality condition

$$* \tilde{G} = Re\mathcal{M} * \tilde{F} - Im\mathcal{M}\tilde{F} \tag{C.22}$$

$$\tilde{G} = Re\mathcal{M}\tilde{F} + Im\mathcal{M} * \tilde{F} \tag{C.23}$$

as an equation of motion for G_A. After elimination of G_A, we obtain

$$\begin{aligned} \mathcal{L}_{F^A} &= +\frac{1}{2}Im\mathcal{M}_{AB}\tilde{F}^A \wedge *\tilde{F}^B + \frac{1}{2}Re\mathcal{M}_{AB}\tilde{F}^A \wedge \tilde{F}^B \\ &\quad +\frac{1}{2}\tilde{e}_A C_2 \wedge \left(F^A + \tilde{F}^A\right). \end{aligned} \tag{C.24}$$

It can be checked that this is compatible with the new component $(2,6)$ of the equation of motion of $\hat{C}_2$ (2.75) which reads now

$$e^{\hat{\phi}}\mathcal{K}d * dC_2 = -\left(F^A\tilde{e}_A - G_A\tilde{m}^A\right) + d\rho_i \wedge db^i. \tag{C.25}$$

After the Weyl rescaling of the volume and the rotation of v^i, the whole action is

$$\begin{aligned} S_{IIB}^{(4)} = & \int -\frac{1}{2}R *\mathbf{1} - g_{ab}dz^a \wedge *d\bar{z}^b - g_{ij}dt^i \wedge *d\bar{t}^j - d\phi \wedge *d\phi \\ & -\frac{1}{4}e^{-4\phi}dB_2 \wedge *dB_2 - \frac{1}{2}e^{-2\phi}\mathcal{K}\left(dC_2 - ldB_2\right) \wedge *\left(dC_2 - ldB_2\right) \\ & -\frac{1}{2}\mathcal{K}e^{2\phi}dl \wedge *dl - 2\mathcal{K}e^{2\phi}g_{ij}\left(dc^i - ldb^i\right) \wedge *\left(dc^j - ldb^j\right) \\ & -\frac{e^{2\phi}}{8\mathcal{K}}g^{-1\,ij}\left(d\rho_i - \mathcal{K}_{ikl}c^k db^l\right) \wedge *\left(d\rho_j - \mathcal{K}_{jmn}c^m db^n\right) \\ & +\left(db^i \wedge C_2 + c^i dB_2\right) \wedge \left(d\rho_i - \mathcal{K}_{ijk}c^j db^k\right) + \frac{1}{2}\mathcal{K}_{ijk}c^i c^j dB_2 \wedge db^k \\ & +\frac{1}{2}Re\mathcal{M}_{AB}\tilde{F}^A \wedge \tilde{F}^B + \frac{1}{2}Im\mathcal{M}_{AB}\tilde{F}^A \wedge *\tilde{F}^B + \frac{1}{2}\tilde{e}_A\left(F^A + \tilde{F}^A\right) \wedge C_2 \\ & +\frac{1}{2}e^{4\phi}\left(l^2 + \frac{e^{-2\phi}}{2\mathcal{K}}\right)\left(\tilde{e} - \mathcal{M}\tilde{m}\right)_A Im\mathcal{M}^{-1AB}\left(\tilde{e} - \bar{\mathcal{M}}\tilde{m}\right)_B *\mathbf{1}\,. \end{aligned} \tag{C.26}$$

We again want to dualize the 2-form C_2 but it has become massive with a mass involving only the magnetic fluxes. This is one of the reasons why the study of the magnetic fluxes is more involved and will not be addressed in this thesis. From now on we consider only the electric fluxes. C_2 and B_2 are thus massless and can be dualized to the scalars h_1 and h_2. We add first

$$+dC_2 \wedge dh_1 \tag{C.27}$$

and the Lagrangian for C_2 is

$$\begin{aligned}\mathcal{L}_{C_2} &= -\frac{1}{2}e^{-2\phi}\mathcal{K}\left(dC_2 - l dB_2\right)\wedge *\left(dC_2 - l dB_2\right) \\ &\quad -b^i dC_2 \wedge d\rho_i + \tilde{e}_A F^A \wedge C_2 + dC_2 \wedge dh_1 .\end{aligned} \tag{C.28}$$

We eliminate dC_2 with its equation of motion and we find

$$\begin{aligned}\mathcal{L}_{C_2} &= -\frac{1}{2\mathcal{K}}e^{2\phi}\left(dh_1 - b^i d\rho_i - \tilde{e}_A V^A\right)\wedge *\left(dh_1 - b^j d\rho_j - \tilde{e}_A V^A\right) \\ &\quad + l dB_2 \wedge \left(dh_1 - b^i d\rho_i - \tilde{e}_A V^A\right).\end{aligned} \tag{C.29}$$

Repeating the same procedure with B_2, we obtain the action for type IIB supergravity on a Calabi-Yau manifold with electric NS fluxes

$$\begin{aligned}S^{(4)}_{IIB} &= \int -\frac{1}{2}R *\mathbf{1} - g_{ab}dz^a \wedge *d\bar{z}^b - g_{ij}dt^i \wedge *d\bar{t}^j - d\phi \wedge *d\phi \\ &\quad -\frac{e^{2\phi}}{8\mathcal{K}}g^{-1\,ij}\left(d\rho_i - \mathcal{K}_{ikl}c^k db^l\right)\wedge *\left(d\rho_j - \mathcal{K}_{jmn}c^m db^n\right) \\ &\quad -2\mathcal{K}e^{2\phi}g_{ij}\left(dc^i - l db^i\right)\wedge *\left(dc^j - l db^j\right) - \frac{1}{2}\mathcal{K}e^{2\phi}dl \wedge *dl \\ &\quad -\frac{1}{2\mathcal{K}}e^{2\phi}\left(dh_1 - b^i d\rho_i - \tilde{e}_A V^A\right)\wedge *\left(dh_1 - b^j d\rho_j - \tilde{e}_A V^A\right) \\ &\quad -\mathrm{e}^{4\phi}D\tilde{h}\wedge *D\tilde{h} \\ &\quad +\frac{1}{2}Re\mathcal{M}_{AB}F^A \wedge F^B + \frac{1}{2}Im\mathcal{M}_{AB}F^A \wedge *F^B \\ &\quad +\frac{1}{2}e^{4\phi}\left(l^2 + \frac{e^{-2\phi}}{2\mathcal{K}}\right)\tilde{e}_A Im\mathcal{M}^{-1AB}\tilde{e}_B *\mathbf{1}\end{aligned} \tag{C.30}$$

with

$$D\tilde{h} = dh_2 + l dh_1 + (c^i - l b^i)d\rho_i - l\tilde{e}_A V^A - \frac{1}{2}\mathcal{K}_{ijk}c^i c^j db^k . \tag{C.31}$$

Applying the map described in section 2.4, we find

$$\begin{aligned}S^{(4)}_{IIB} &= \int -\frac{1}{2}R *\mathbf{1} - g_{ab}dz^a \wedge *d\bar{z}^b - h_{uv}Dq^u \wedge *Dq^v - V_{IIB} *\mathbf{1} \\ &\quad +\frac{1}{2}Re\mathcal{M}_{AB}F^A \wedge F^B + \frac{1}{2}Im\mathcal{M}_{AB}F^A \wedge *F^B ,\end{aligned} \tag{C.32}$$

where the quaternionic metric is given by

$$\begin{aligned}h_{uv}Dq^u \wedge *Dq^v &= g_{ij}dt^i \wedge *d\bar{t}^j + d\phi \wedge *d\phi \\ &\quad -\frac{1}{2}e^{2\phi}Im\mathcal{N}^{-1\,IJ}\left(D\tilde{\xi}_I + \mathcal{N}_{IK}D\xi^K\right)\wedge *\left(D\tilde{\xi}_J + \bar{\mathcal{N}}_{JL}D\xi^L\right) \\ &\quad +\frac{1}{4}e^{4\phi}\left(Da + (\tilde{\xi}_I D\xi^I - \xi^I D\tilde{\xi}_I)\right)\wedge *\left(Da + (\tilde{\xi}_I D\xi^I - \xi^I D\tilde{\xi}_I)\right) ,\end{aligned} \tag{C.33}$$

while the potential reads

$$V_{IIB} = -\frac{1}{2}\,e^{4\phi}\left(l^2 + \frac{e^{-2\phi}}{2\mathcal{K}}\right)\tilde{e}_A\left[(Im\mathcal{M})^{-1}\right]^{AB}\tilde{e}_B . \tag{C.34}$$

The presence of the electric fluxes has gauged some of the isometries of the hyperscalars as can be seen from the covariant derivatives

$$Da = da - \xi^0 \tilde{e}_A V^A \ , \qquad D\tilde{\xi}_0 = d\tilde{\xi}_0 + \tilde{e}_A V^A \ , \qquad D\tilde{\xi}_i = d\tilde{\xi}_i \ , \qquad D\xi^I = d\xi^I \ . \tag{C.35}$$

Appendix D

G-structures

In this section we assemble a few facts about G-structures as taken from the mathematical literature where one also finds the proofs omitted here. (See, for example, [39–41,43,44,91,100].) We concentrate on the example of manifolds with $SU(3)$-structure.

D.0.1 Almost Hermitian manifolds

Before discussing G-structures in general, let us recall the definition of an almost Hermitian manifold. This allows us to introduce useful concepts, and, as we subsequently will see, provides us with a classic example of a G-structure.

A manifold of real dimension $2n$ is called *almost complex* if it admits a globally defined tensor field $J_m{}^n$ which obeys

$$J_m{}^p J_p{}^n = -\delta_m{}^n \ . \tag{D.1}$$

A metric g_{mn} on such a manifold is called Hermitian if it satisfies

$$J_m{}^p J_n{}^r g_{pr} = g_{mn} \ . \tag{D.2}$$

An almost complex manifold endowed with a Hermitian metric is called an *almost Hermitian manifold*. The relation (D.2) implies that $J_{mn} = J_m{}^p g_{pn}$ is a non-degenerate 2-form which is called *the fundamental form*.

On any even dimensional manifold one can locally introduce complex coordinates. However, complex manifolds have to satisfy in addition that, first, the introduction of complex coordinates on different patches is consistent, and second that the transition functions between different patches are holomorphic functions of the complex coordinates. The first condition corresponds to the existence of an almost complex structure. The second condition is an integrability condition, implying that there are coordinates such that the almost complex structure takes the form

$$J = \begin{pmatrix} i\mathbf{1_{n\times n}} & 0 \\ 0 & -i\mathbf{1_{n\times n}} \end{pmatrix} \ . \tag{D.3}$$

The integrability condition is satisfied if and only if the Nijenhuis tensor $N_{mn}{}^p$ vanishes. It is defined as

$$\begin{aligned} N_{mn}{}^p &= J_m{}^q \left(\partial_q J_n{}^p - \partial_n J_q{}^p\right) - J_n{}^q \left(\partial_q J_m{}^p - \partial_m J_q{}^p\right) \\ &= J_m{}^q \left(\nabla_q J_n{}^p - \nabla_n J_q{}^p\right) - J_n{}^q \left(\nabla_q J_m{}^p - \nabla_m J_q{}^p\right) \ , \end{aligned} \tag{D.4}$$

where ∇ denotes the covariant derivative with respect to the Levi–Civita connection.

One can also consider an even stronger condition where $\nabla_m J_{np} = 0$. This implies $N_{mn}{}^p = 0$ but in addition that $\mathrm{d}J = 0$ and means we have a *Kähler manifold*. In particular, it implies that the holonomy of the Levi–Civita connection ∇ is $U(n)$.

Even if there is no coordinate system where it can be put in the form (D.3), any almost complex structure obeying (D.1) has eigenvalues $\pm i$. Thus even for non-integrable almost complex

structures one can define the projection operators

$$(P^{\pm})_m{}^n = \frac{1}{2}(\delta_m^n \mp iJ_m{}^n) \ , \tag{D.5}$$

which project onto the two eigenspaces, and satisfy

$$P^{\pm}P^{\pm} = P^{\pm} \ , \qquad P^+P^- = 0 \ . \tag{D.6}$$

On an almost complex manifold one can define (p, q) projected components $\omega^{p,q}$ of a real $(p+q)$-form ω^{p+q} by using (D.5)

$$\omega^{p,q}_{m_1\ldots m_{p+q}} = (P^+)_{m_1}{}^{n_1} \ldots (P^+)_{m_p}{}^{n_p}(P^-)_{m_{p+1}}{}^{n_{p+1}} \ldots (P^-)_{m_{p+q}}{}^{n_{p+q}}\omega^{p+q}_{n_1\ldots n_{p+q}} \ . \tag{D.7}$$

Furthermore, a real $(p + q)$-form is of the type (p, q) if it satisfies

$$\omega_{m_1\ldots m_p n_1\ldots n_q} = (P^+)_{m_1}{}^{r_1} \ldots (P^+)_{m_q}{}^{r_p}(P^-)_{n_1}{}^{s_1} \ldots (P^-)_{n_q}{}^{s_q}\omega_{r_1\ldots s_q} \ . \tag{D.8}$$

In analogy with complex manifolds we denote the projections on the subspace of eigenvalue $+i$ with an unbarred index α and the projection on the subspace of eigenvalue $-i$ with a barred index $\bar{\alpha}$. For example the hermitian metric of an almost Hermitian manifold is of type $(1, 1)$ and has one barred and one unbarred index. Thus, raising and lowering indices using this hermitian metric converts holomorphic indices into anti-holomorphic ones and vice versa. Moreover the contraction of a holomorphic and an anti-holomorphic index vanishes, i.e. given V_m which is of type $(1, 0)$ and W^n which is of type $(0, 1)$, the product V_mW^m is zero. Similarly, on an almost hermitian manifold of real dimension $2n$ forms of type $(p, 0)$ vanish for $p > n$. Finally, derivatives of (p, q)-forms pick up extra pieces compared to complex manifolds precisely because J is not constant. One finds [91]

$$d\omega^{(p,q)} = (d\omega)^{(p-1,q+2)} + (d\omega)^{(p,q+1)} + (d\omega)^{(p+1,q)} + (d\omega)^{(p+2,q-1)} \ . \tag{D.9}$$

D.0.2 G-structures and G-invariant tensors

An orthonormal frame on a d-dimensional Riemannian manifold M is given by a basis of vectors e_i, with $i = 1, \ldots, d$, satisfying $e_i^m e_j^n g_{mn} = \delta_{ij}$. The set of all orthonormal frames is known as the frame bundle. In general, the structure group of the frame bundle is the group of rotations $O(d)$ (or $SO(d)$ is M is orientable). The manifold has a G-structure if the structure group of the frame bundle is not completely general but can be reduced to $G \subset O(d)$. For example, in the case of an almost Hermitian manifold of dimension $d = 2n$, it turns out one can always introduce a complex frame and as a result the structure group reduces to $U(n)$.

An alternative and sometimes more convenient way to define G-structures is via G-invariant tensors, or, if M is spin, G-invariant spinors. A non-vanishing, globally defined tensor or spinor ξ is G-invariant if it is invariant under $G \subset O(d)$ rotations of the orthonormal frame. In the case of almost Hermitian structure, the two-form J is an $U(n)$-invariant tensor. Since the invariant tensor ξ is globally defined, by considering the set of frames for which ξ takes the same fixed form, one can see that the structure group of the frame bundle must then reduce to G (or a subgroup of G). Thus the existence of ξ implies we have a G-structure. Typically, the converse is also true. Recall that, relative to an orthonormal frame, tensors of a given type form the vector space for a given representation of $O(d)$ (or $Spin(d)$ for spinors). If the structure group of the frame bundle is reduced to $G \subset O(d)$, this representation can be decomposed into irreducible representations of G. In the case of almost complex manifolds, this corresponds to the decomposition under the $P^{\pm}$ projections (D.5). Typically there will be some tensor or spinor that will have a component in this decomposition which is invariant under G. The corresponding vector bundle of this component must be trivial, and thus will admit a globally defined non-vanishing section ξ. In other words, we have a globally defined non-vanishing G-invariant tensor or spinor.

To see this in more detail in the almost complex structure example, recall that we had a globally defined fundamental two-form J. Let us specialize for definiteness to a six-manifold, though the argument is quite general. Two-forms are in the adjoint representation **15** of $SO(6)$ which decomposes under $U(3)$ as

$$\mathbf{15} = \mathbf{1} + \mathbf{8} + (\mathbf{3} + \bar{\mathbf{3}}) \ . \tag{D.10}$$

There is indeed a singlet in the decomposition and so given a $U(3)$-structure we necessarily have a globally defined invariant two-form, which is precisely the fundamental two-form J. Conversely, given a metric and a non-degenerate two-form J, we have an almost Hermitian manifold and consequently a $U(3)$-structure.

In this paper we are interested in $SU(3)$-structure. In this case we find two invariant tensors. First we have the fundamental form J as above. In addition, we find an invariant complex three-form Ω. Three-forms are in the **20** representation of $SO(6)$, giving two singlets in the decomposition under $SU(3)$,

$$\begin{aligned} \mathbf{15} &= \mathbf{1} + \mathbf{8} + \mathbf{3} + \bar{\mathbf{3}} \quad \Rightarrow \quad J \ , \\ \mathbf{20} &= \mathbf{1} + \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{6} + \bar{\mathbf{6}} \quad \Rightarrow \quad \Omega = \Omega^+ + \mathrm{i}\Omega^- \ . \end{aligned} \tag{D.11}$$

In addition, since there is no singlet in the decomposition of a five-form, one finds that

$$J \wedge \Omega = 0 \ . \tag{D.12}$$

Similarly, a six-form is a singlet of $SU(3)$, so we also must have that $J \wedge J \wedge J$ is proportional to $\Omega \wedge \bar{\Omega}$. The usual convention is to set

$$J \wedge J \wedge J = \frac{3\mathrm{i}}{4} \Omega \wedge \bar{\Omega} \ , \tag{D.13}$$

Conversely, a non-degenerate J and Ω satisfying (D.12) and (D.13) implies that M has $SU(3)$-structure. Note that, unlike the $U(n)$ case, the metric need not be specified in addition; the existence of J and Ω is sufficient [61]. Essentially this is because, without the presence of a metric, Ω defines an almost complex structure, and J an almost symplectic structure. Treating J as the fundamental form, it is then a familiar result on almost Hermitian manifolds that the existence of an almost complex structure and a fundamental form allow one to construct a Hermitian metric.

We can similarly ask what happens to spinors for a structure group $SU(3)$. In this case we have the isomorphism $Spin(6) \cong SU(4)$ and the four-dimensional spinor representation decomposes as

$$\mathbf{4} = \mathbf{1} + \mathbf{3} \quad \Rightarrow \quad \eta \ . \tag{D.14}$$

We find one singlet in the decomposition, implying the existence of a globally defined invariant spinor η. Again, the converse is also true. A metric and a globally defined spinor η implies that M has $SU(3)$-structure.

D.0.3 Intrinsic torsion

One would like to have some classification of G-structures. In particular, one would like a generalization of the notion of a Kähler manifold where the holonomy of the Levi–Civita connection reduces to $U(n)$. Such a classification exists in terms of the *intrinsic torsion*. Let us start by recalling the definition of torsion and contorsion on a Riemannian manifold (M, g).

Given any metric compatible connection ∇' on (M, g), i.e. one satisfying $\nabla'_m g_{np} = 0$, one can define the Riemann curvature tensor and the torsion tensor as follows

$$[\nabla'_m, \nabla'_n] V_p = -R_{mnp}{}^q V_q - 2T_{mn}{}^r \nabla'_r V_p \ , \tag{D.15}$$

where V is an arbitrary vector field. The Levi-Civita connection is the unique torsionless connection compatible with the metric and is given by the usual expression in terms of Christoffel symbols $\Gamma_{mn}{}^{p} = \Gamma_{nm}{}^{p}$. Let us denote by ∇ the covariant derivative with respect to the Levi-Civita connection while a connection with torsion is denoted by $\nabla^{(T)}$. Any metric compatible connection can be written in terms of the Levi-Civita connection

$$\nabla^{(T)} = \nabla - \kappa \ , \tag{D.16}$$

where $\kappa_{mn}{}^{p}$ is the contorsion tensor. Metric compatibility implies

$$\kappa_{mnp} = -\kappa_{mpn} \ , \quad \text{where} \quad \kappa_{mnp} = \kappa_{mn}{}^{r} g_{rp} \ . \tag{D.17}$$

Inserting (D.17) into (D.15) one finds a one-to-one correspondence between the torsion and the contorsion

$$\begin{aligned} T_{mn}{}^{p} &= \frac{1}{2}(\kappa_{mn}{}^{p} - \kappa_{nm}{}^{p}) \equiv \kappa_{[mn]}{}^{p} \ , \\ \kappa_{mnp} &= T_{mnp} + T_{pmn} + T_{pnm} \ . \end{aligned} \tag{D.18}$$

These relations tell us that given a torsion tensor T there exist a unique connection $\nabla^{(T)}$ whose torsion is precisely T.

Now suppose M has a G-structure. In general the Levi-Civita connection does not preserve the G-invariant tensors (or spinor) ξ. In other words, $\nabla\xi \neq 0$. However, one can show [41], that there always exist some other connection $\nabla^{(T)}$ which is compatible with the G structure so that

$$\nabla^{(T)}\xi = 0 \ . \tag{D.19}$$

Thus for instance, on an almost Hermitian manifold one can always find $\nabla^{(T)}$ such that $\nabla^{(T)}J = 0$. On a manifold with $SU(3)$-structure, it means we can always find $\nabla^{(T)}$ such that both $\nabla^{(T)}J = 0$ and $\nabla^{(T)}\Omega = 0$. Since the existence of an $SU(3)$-structure is also equivalent to the existence of an invariant spinor η, this is equivalent to the condition $\nabla^{(T)}\eta = 0$.

Let κ be the contorsion tensor corresponding to $\nabla^{(T)}$. From the symmetries (D.17), we see that κ is an element of $\Lambda^1 \otimes \Lambda^2$ where Λ^n is the space of n-forms. Alternatively, since $\Lambda^2 \cong so(d)$, it is more natural to think of $\kappa_{mn}{}^{p}$ as one-form with values in the Lie-algebra $so(d)$ that is $\Lambda^1 \otimes so(d)$. Given the existence of a G-structure, we can decompose $so(d)$ into a part in the Lie algebra g of $G \subset SO(d)$ and an orthogonal piece $g^\perp = so(d)/g$. The contorsion κ splits accordingly into

$$\kappa = \kappa^0 + \kappa^g \ , \tag{D.20}$$

where κ^0 is the part in $\Lambda^1 \otimes g^\perp$. Since an invariant tensor (or spinor) ξ is fixed under G rotations, that action of g on ξ vanishes and we have, by definition,

$$\nabla^{(T)}\xi = \left(\nabla - \kappa^0 - \kappa^g\right)\xi = \left(\nabla - \kappa^0\right)\xi = 0 \ . \tag{D.21}$$

Thus, any two G-compatible connections must differ by a piece proportional to κ^g and they have a common term κ^0 in $\Lambda^1 \otimes g^\perp$ called the "intrinsic contorsion". Recall that there is an isomorphism (D.18) between κ and T. It is more conventional in the mathematics literature to define the corresponding torsion

$$T^0_{mn}{}^{p} = \kappa^0_{[mn]}{}^{p} \in \Lambda^1 \otimes g^\perp \ , \tag{D.22}$$

known as the *intrinsic torsion*.

From the relation (D.21) it is clear that the intrinsic contorsion, or equivalently torsion, is independent of the choice of G-compatible connection. Basically it is a measure of the degree to which $\nabla\xi$ fails to vanish and as such is a measure solely of the G-structure itself. Furthermore, one can decompose κ^0 into irreducible G representations. This provides a classification of G-structures in terms of which representations appear in the decomposition. In particular, in the

special case where κ^0 vanishes so that $\nabla\xi = 0$, one says that the structure is "torsion-free". For an almost Hermitian structure this is equivalent to requiring that the manifold is complex and Kähler. In particular, it implies that the holonomy of the Levi–Civita connection is contained in G.

Let us consider the decomposition of T^0 in the case of $SU(3)$-structure. The relevant representations are

$$\Lambda^1 \sim \mathbf{3} \oplus \bar{\mathbf{3}} , \qquad g \sim \mathbf{8} , \qquad g^\perp \sim \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} . \tag{D.23}$$

Thus the intrinsic torsion, which is an element of $\Lambda^1 \otimes su(3)^\perp$, can be decomposed into the following $SU(3)$ representations

$$\begin{aligned} \Lambda^1 \otimes su(3)^\perp &= (\mathbf{3} \oplus \bar{\mathbf{3}}) \otimes (\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}) \\ &= (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{8} \oplus \mathbf{8}) \oplus (\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus (\mathbf{3} \oplus \bar{\mathbf{3}}) \oplus (\mathbf{3} \oplus \bar{\mathbf{3}})' . \end{aligned} \tag{D.24}$$

The terms in parentheses on the second line correspond precisely to the five classes $\mathcal{W}_1, \dots, \mathcal{W}_5$ presented in table 3.1. We label the component of T^0 in each class by $T_1, \dots, T_5$.

In the case of $SU(3)$-structure, each component T_i can be related to a particular component in the $SU(3)$ decomposition of dJ and dΩ. From (D.21), we have

$$\begin{aligned} dJ_{mnp} &= 6T^0_{[mn}{}^r J_{r|p]} , \\ d\Omega_{mnpq} &= 12T^0_{[mn}{}^r \Omega_{r|pq]} . \end{aligned} \tag{D.25}$$

Since J and Ω are $SU(3)$ singlets, dJ and $d\Omega$ are both elements of $\Lambda^1 \otimes su(3)^\perp$. Put another way, the contractions with J and Ω in (D.25) simply project onto different $SU(3)$ representations of T^0. We can see which representations appear simply by decomposing the real three-form dJ and complex four-form $d\Omega$ under $SU(3)$. We have,

$$\begin{aligned} & dJ = \left[(dJ)^{3,0} + (dJ)^{0,3}\right] + \left[(dJ)^{2,1}_0 + (dJ)^{1,2}_0\right] + \left[(dJ)^{1,0} + (dJ)^{0,1}\right] , \\ & \mathbf{20} = (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus (\mathbf{3} \oplus \bar{\mathbf{3}}) , \end{aligned} \tag{D.26}$$

and

$$\begin{aligned} d\Omega &= (d\Omega)^{3,1} + (d\Omega)^{2,2}_0 + (d\Omega)^{0,0} , \\ \mathbf{24} &= (\mathbf{3} \oplus \bar{\mathbf{3}})' \oplus (\mathbf{8} \oplus \mathbf{8}) \oplus (\mathbf{1} \oplus \mathbf{1}) . \end{aligned} \tag{D.27}$$

The superscripts in the decomposition of dJ and $d\Omega$ refer to the (p,q)-type of the form. The 0 subscript refers to the irreducible $SU(3)$ representation where the trace part, proportional to J^n has been removed. Thus in particular, the traceless parts $(dJ)^{2,1}_0$ and $(d\Omega)^{2,2}_0$ satisfy $J \wedge (dJ)^{2,1}_0 = 0$ and $J \wedge (d\Omega)^{2,2}_0 = 0$ respectively. The trace parts on the other hand, have the form $(dJ)^{1,0} = \alpha \wedge J$ and $(d\Omega)^{0,0} = \beta J \wedge J$, with $\alpha \sim *(J \wedge dJ)$ and $\beta \sim *(J \wedge d\Omega)$ respectively. Note that a generic complex four-form has 30 components. However, since Ω is a $(3,0)$-form, from (D.9) we see that $d\Omega$ has no $(1,3)$ part, and so only has 24 components. Comparing (D.26) and (D.27) with (D.24) we see that

$$dJ \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 , \quad d\Omega \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_5 , \tag{D.28}$$

and as advertised, dJ and $d\Omega$ together include all the components T_i. Explicit expressions for some of these relations are given above in (3.26) and (3.29). Note that the singlet component T_1 can be expressed either in terms of $(dJ)^{0,3}$, corresponding to $\Omega \wedge dJ$ or in terms of $(d\Omega)^{0,0}$ corresponding to $J \wedge d\Omega$. This is simply a result of the relation (D.12) which implies that $\Omega \wedge dJ = J \wedge d\Omega$.

Appendix E

The Ricci scalar of half-flat manifolds

The Riemann curvature tensor is defined as

$$R_{mnp}{}^{q} = \partial_m \phi_{np}{}^{q} - \partial_n \phi_{mp}{}^{q} - \phi_{mp}{}^{r} \phi_{nr}{}^{q} + \phi_{np}{}^{r} \phi_{mr}{}^{q}, \tag{E.1}$$

where ϕ denotes a general connection that contains two contributions $\phi_{mn}{}^{p} = \Gamma_{mn}{}^{p} + \kappa_{mn}{}^{p}$ where $\Gamma_{mn}{}^{p} = \Gamma_{nm}{}^{p}$ denote the Christoffel symbols and $\kappa_{mn}{}^{p}$ is the contorsion, out of which we define the torsion

$$\begin{aligned} T_{mn}{}^{p} &= \frac{1}{2}(\kappa_{mn}{}^{p} - \kappa_{nm}{}^{p}) && \text{(E.2)} \\ \kappa_{mnp} &= T_{mnp} + T_{pmn} + T_{pnm}. && \text{(E.3)} \end{aligned}$$

For the Ricci tensor we use $R_{np} = R_{nmp}{}^{m}$. The simplest way to derive the Ricci scalar for the manifold considered in section 3.2.1 is by using the integrability condition one can derive from the Killing spinor equation (3.14).

$$R^{(T)}_{mnpq} \Gamma^{pq} \eta = 0, \tag{E.4}$$

where the Riemann tensor of the connection with torsion is given by (E.1)

$$R^{(T)}_{mnpq} = R(\Gamma)_{mnpq} + \nabla_m \kappa_{npq} - \nabla_n \kappa_{mpq} - \kappa_{mp}{}^{r} \kappa_{nrq} + \kappa_{np}{}^{r} \kappa_{mrq} \ . \tag{E.5}$$

Here $R(\Gamma)_{mnpq}$ represents the usual Riemann tensor for the Levi-Civita connection and the covariant derivatives are again with respect to the Levi-Civita connection. For definiteness we choose the solution of the Killing spinor equation (3.14) to be a Majorana spinor.[1] Multiplying (E.4) by Γ^n and summing over n one obtains

$$R^{(T)}_{mnpq} \Gamma^{npq} \eta - 2R^{(T)}_{mn} \Gamma^{n} \eta = 0 \ . \tag{E.6}$$

Contracting from the left with $\eta^\dagger \Gamma^m$ and using the conventions for the Majorana spinors (A.33) one derives

$$2R^{(T)} = R^{(T)}_{mnpq} \eta^\dagger \Gamma^{mnpq} \eta \ . \tag{E.7}$$

where $R^{(T)}$ represents the Ricci scalar which can be defined from the Riemann tensor (E.5). Expressing $R^{(T)}_{mnpq}$ in terms of $R(\Gamma)_{mnpq}$ from (E.5), using the Bianchi identity $R(\Gamma)_{m[npq]} = 0$ and the fact that the contorsion is traceless $\kappa_{mn}{}^{m} = \kappa^{m}{}_{mn} = 0$ which holds for half flat manifolds one can derive the formula for the Ricci scalar of the Levi-Civita connection

$$R = -\kappa_{mnp} \kappa^{npm} - \frac{1}{2} \epsilon^{mnpqrs} (\nabla_m \kappa_{npq} - \kappa_{mp}{}^{l} \kappa_{nlq}) J_{rs} \ . \tag{E.8}$$

In order to simplify the formulas we evaluate (E.8) term by term. The strategy will be to express first the contorsion κ in terms of the torsion T (E.3) and then go to complex indices

[1] The results are independent of the choice of the spinor, but the derivations may be more involved.

splitting the torsion in its component parts $T_{1\oplus 2}$ and T_3 which are of definite type with respect to the almost complex structure J.

The first term can be written as

$$A \equiv -\kappa_{mnp}\kappa^{npm} = -(T_{mnp} + T_{pmn} + T_{pnm})T^{npm} = T_{mnp}T^{mnp} - 2T_{mnp}T^{npm}. \tag{E.9}$$

Using (3.27) and (3.29) one sees that the first two indices of T are of the same type and thus one has

$$A = (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} - 2(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} + (T_3)_{\alpha\beta\bar{\gamma}}(T_3)^{\alpha\beta\bar{\gamma}} + c.c.\ , \tag{E.10}$$

where *c.c.* denotes complex conjugation.

The second term can be computed if one takes into account that the 4 dimensional effective action appears after one integrates the 10 dimensional action over the internal space, in this case $\hat{Y}$. Thus the second term in (E.8) can be integrated by parts to give[2]

$$B \equiv -\frac{1}{2}\epsilon^{mnpqrs}(\nabla_m\kappa_{npq})J_{rs} \sim \frac{1}{2}\epsilon^{mnpqrs}\kappa_{npq}\nabla_m J_{rs}. \tag{E.11}$$

Using (3.15) and (D.18) we obtain after going to complex indices

$$\begin{aligned} B &= -\epsilon^{mnpqrs}T_{mnp}T_{qr}{}^{t}J_{ts} \\ &= -\epsilon^{\alpha\beta\gamma\bar{\alpha}\bar{\beta}\bar{\gamma}}(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\bar{\alpha}\bar{\beta}}{}^{\delta}J_{\delta\bar{\gamma}} - \epsilon^{\alpha\beta\bar{\gamma}\bar{\alpha}\bar{\beta}\gamma}(T_3)_{\alpha\beta\bar{\gamma}}(T_3)_{\bar{\alpha}\bar{\beta}}{}^{\bar{\delta}}J_{\bar{\delta}\gamma} + c.c.\ . \end{aligned} \tag{E.12}$$

The 6 dimensional ϵ symbol splits as

$$\epsilon^{\alpha\beta\gamma\bar{\alpha}\bar{\beta}\bar{\gamma}} = -i\epsilon^{\alpha\beta\gamma}\epsilon^{\bar{\alpha}\bar{\beta}\bar{\gamma}}\ , \tag{E.13}$$

and after some algebra involving the 3 dimensional ϵ symbol one finds

$$B = -2(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} - 4(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} - 2(T_3)_{\alpha\beta\bar{\gamma}}(T_3)^{\alpha\beta\bar{\gamma}} + c.c.\ . \tag{E.14}$$

In the same way one obtains for the last term

$$C \equiv \frac{1}{2}\epsilon^{mnpqrs}\kappa_{mp}{}^{t}\kappa_{ntq}J_{rs} = 2(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} + 2(T_3)_{\alpha\beta\bar{\gamma}}(T_3)^{\alpha\beta\bar{\gamma}} + c.c.\ . \tag{E.15}$$

Collecting the results from (E.10), (E.14) and (E.15) the formula for the Ricci scalar (E.8) becomes

$$R = (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} - 6(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} + (T_3)_{\alpha\beta\bar{\gamma}}(T_3)^{\alpha\beta\bar{\gamma}} + c.c.\ . \tag{E.16}$$

The first two terms in the above expression can be straightforwardly computed using (3.27), (3.38) and

$$(T_{1\oplus 2})_{\alpha\beta\gamma} = \frac{e_i}{4||\Omega||^2}(\tilde{\omega}^i)_{\alpha\beta\bar{\alpha}\bar{\beta}}\Omega^{\bar{\alpha}\bar{\beta}}{}_{\gamma}. \tag{E.17}$$

After a little algebra we find

$$\begin{aligned} (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} &= \frac{e_ie_j}{8||\Omega||^2}(\tilde{\omega}^i)_{\alpha\beta\bar{\alpha}\bar{\beta}}(\tilde{\omega}^j)^{\alpha\beta\bar{\alpha}\bar{\beta}} \\ (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} &= -\frac{e_ie_j}{8||\Omega||^2}(\tilde{\omega}^i)_{\alpha\beta\bar{\alpha}\bar{\beta}}(\tilde{\omega}^j)^{\alpha\beta\bar{\alpha}\bar{\beta}} - \frac{e_ie_j}{4||\Omega||^2}(*\tilde{\omega}^i)_{\alpha\bar{\beta}}(*\tilde{\omega}^j)^{\bar{\beta}\alpha} + \frac{(e_iv^i)^2}{4||\Omega||^2\mathcal{K}^2}\ . \end{aligned} \tag{E.18}$$

In order to obtain the above expressions we have used (B.20) and [92]

[2]Strictly speaking in 10 dimensions the Ricci scalar comes multiplied with a dilaton factor (3.48). However in all that we are doing we consider that the dilaton is constant over the internal space so it still make sense to speak about integration by parts without introducing additional factors with derivatives of the dilaton.

$$(\tilde{\omega}^i)_{\alpha\beta\bar{\beta}}{}^{\beta} = -ig_{\alpha\bar{\beta}}(*\tilde{\omega}^i)_{\gamma\bar{\gamma}}g^{\gamma\bar{\gamma}} + i(*\tilde{\omega}^i)_{\alpha\bar{\beta}} \tag{E.19}$$

$$(\tilde{\omega}^i)_{\alpha\beta}{}^{\alpha\beta} = \frac{2v^i}{\mathcal{K}}. \tag{E.20}$$

Integrating (E.18) over $\hat{Y}$ we obtain

$$\begin{aligned}
\int_{\hat{Y}} (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} &= \frac{e_i e_j g^{ij}}{8||\Omega||^2\mathcal{K}} \,, \\
\int_{\hat{Y}} (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} &= -\frac{e_i e_j g^{ij}}{16||\Omega||^2\mathcal{K}} + \frac{(e_i v^i)^2}{4||\Omega||^2\mathcal{K}} \,.
\end{aligned} \tag{E.21}$$

Finally, we have to compute the third term in (E.16). For this we note that, since dJ is in $W_1^- + W_2^- + W_3$, the torsions T_{12} and T_3 both appear in (3.46). Projecting (3.46) on the $(3,0)$ component, we find a relation between T_{12} and $\beta^0|_{(3,0)}$, which using the explicit expression for the torsion (E.17) can be written

$$\beta^0_{\alpha\beta\gamma} = -\frac{i}{\mathcal{K}||\Omega||^2}\Omega_{\alpha\beta\gamma}. \tag{E.22}$$

The $(2,1)$ component gives T_3 in terms of the $(2,1)$ part of β^0

$$(T_3)_{\alpha\beta\bar{\gamma}} = -\frac{i}{2}v^i e_i\, \beta^0_{\alpha\beta\bar{\gamma}}. \tag{E.23}$$

To obtain an explicit expression for the third term in (E.16), we compute the following quantity

$$\int_{\hat{Y}} \beta^0 \wedge *\beta^0 = \frac{2}{\mathcal{K}||\Omega||^2} + \int_{\hat{Y}} \beta^0_{\alpha\beta\bar{\gamma}}\beta^{0\,\alpha\beta\bar{\gamma}}. \tag{E.24}$$

On the other hand, the same integral appears in

$$\int \beta^0 \wedge *\beta^0 = -\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00} = \frac{8}{||\Omega||^2\mathcal{K}}. \tag{E.25}$$

The simplest way to see this is by using a mirror symmetry argument. We know that under mirror symmetry the gauge couplings $\mathcal{M}$ and $\mathcal{N}$ are mapped into one another. This also means that $(\mathrm{Im}\,\mathcal{M})^{-1}$ is mapped into $(\mathrm{Im}\,\mathcal{N})^{-1}$ and this matrix is given in (B.86) for a Calabi-Yau space. From here one sees that the element $\left[(\mathrm{Im}\,\mathcal{N})^{-1}\right]^{00}$ is just the inverse volume of the mirror Calabi-Yau space. Using again mirror symmetry and the fact that the Kähler potential of the Kähler moduli (B.81) is mapped into the Kähler potential of the complex structure moduli (B.92) we end up with the RHS of the above equation.

Now we obtain

$$\int_{\hat{Y}} (T_3)_{\alpha\beta\bar{\gamma}}(T_3)^{\alpha\beta\bar{\gamma}} = \frac{3}{2}\,\frac{(e_i v^i)^2}{||\Omega||^2\mathcal{K}} \,. \tag{E.26}$$

Inserting (E.21) and (E.26) into (E.16) and taking into account that all the terms in (E.21) and (E.26) are explicitly real such that the term '*c.c.*' in (E.16) just introduces one more factor of 2, we obtain the final form of the Ricci scalar

$$R = -\frac{1}{8}\, e_i e_j g^{ij}\, \left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00} \,, \tag{E.27}$$

where we have used again (E.25).

Appendix F

Display of torsion and curvature components

The conventional constraints compatible with the assumptions (4.15 – 4.16) and (4.21) are the following:

$$
\begin{array}{rclcrcl}
T_{\gamma}^{\mathrm{C}}{}_{b}{}^{a} &=& 0 & \qquad & T^{\dot{\gamma}}_{\mathrm{C}}{}_{b}{}^{a} &=& 0 \\
T_{\gamma}^{\mathrm{C}}{}_{\beta}^{\mathrm{B}}{}_{\mathrm{A}}^{\alpha} &=& 0 & & T^{\dot{\gamma}}_{\mathrm{C}}{}^{\dot{\beta}}_{\mathrm{B}}{}^{\mathrm{A}}_{\dot{\alpha}} &=& 0 \\
T_{\gamma}^{\mathrm{C}}{}^{\dot{\beta}}_{\mathrm{B}}{}^{\mathrm{A}}_{\dot{\alpha}} &=& 0 & & T^{\gamma}_{\mathrm{C}}{}^{\mathrm{B}}_{\dot{\beta}}{}^{\dot{\alpha}}_{\mathrm{A}} &=& 0 \\
T_{c}{}^{\mathrm{B}}_{\alpha}{}^{\alpha}_{\mathrm{A}} &=& 0 & & T_{c}{}^{\dot{\alpha}}_{\mathrm{B}}{}^{\mathrm{A}}_{\dot{\alpha}} &=& 0
\end{array}
\tag{F.1}
$$

$$T_{cb}{}^{a} = 0\,.$$

There is a particular solution of the Bianchi identities for the torsion and 3–form subject to the constraints (4.15 – 4.18), (4.20 – 4.21) and (F.1), which describes the N-T supergravity multiplet. Besides the constant $T^{\dot{\gamma}}_{\mathrm{C}}{}^{\mathrm{B}}_{\beta}{}^{a}$ and the supercovariant field strength of the graviphotons, $T_{cb}{}^{\mathbf{u}} \doteq F_{cb}{}^{\mathbf{u}}$, the non-zero torsion components corresponding to this solution are then the following:

$$
\begin{array}{rclcrcl}
T_{\gamma}^{\mathrm{C}}{}_{\beta}^{\mathrm{B}}{}^{\mathbf{u}} &=& 4\epsilon_{\gamma\beta}\mathbf{t}^{[\mathrm{CB}]\mathbf{u}}e^{\phi} & \qquad & T^{\dot{\gamma}}_{\mathrm{C}}{}^{\dot{\beta}}_{\mathrm{B}}{}^{\mathbf{u}} &=& 4\epsilon^{\dot{\gamma}\dot{\beta}}\mathbf{t}_{[\mathrm{CB}]}{}^{\mathbf{u}}e^{\phi} \\
T_{\gamma}^{\mathrm{C}}{}_{\beta}^{\mathrm{B}}{}^{\mathrm{A}}_{\dot{\alpha}} &=& q\epsilon_{\gamma\beta}\varepsilon^{\mathrm{CBAF}}\bar{\lambda}_{\mathrm{F}\dot{\alpha}} & & T^{\dot{\gamma}}_{\mathrm{C}}{}^{\dot{\beta}}_{\mathrm{B}}{}^{\alpha}_{\mathrm{A}} &=& q\epsilon^{\dot{\gamma}\dot{\beta}}\varepsilon_{\mathrm{CBAF}}\lambda^{\mathrm{F}\alpha} \\
T_{\gamma}^{\mathrm{C}}{}_{b}{}^{\mathbf{u}} &=& ie^{\phi}(\sigma_b\bar{\lambda}_{\mathrm{A}})_{\gamma}\mathbf{t}^{[\mathrm{AC}]\mathbf{u}} & & T^{\dot{\gamma}}_{\mathrm{C}}{}_{b}{}^{\mathbf{u}} &=& ie^{\phi}(\bar{\sigma}_b\lambda^{\mathrm{A}})^{\dot{\gamma}}\mathbf{t}_{[\mathrm{AC}]}{}^{\mathbf{u}} \\
T_{\gamma}^{\mathrm{C}}{}_{b}{}^{\alpha}_{\mathrm{A}} &=& -2(\sigma_{ba})_{\gamma}{}^{\alpha}U^{a\mathrm{C}}{}_{\mathrm{A}} & & T^{\dot{\gamma}}_{\mathrm{C}}{}_{b}{}^{\mathrm{A}}_{\dot{\alpha}} &=& 2(\bar{\sigma}_{ba})^{\dot{\gamma}}{}_{\dot{\alpha}}U^{a\mathrm{A}}{}_{\mathrm{C}} \\
T_{\gamma}^{\mathrm{C}}{}_{b}{}^{\mathrm{A}}_{\dot{\alpha}} &=& \frac{i}{2}(\sigma_b\bar{\sigma}^{dc})_{\gamma\dot{\alpha}}F_{dc}{}^{[\mathrm{CA}]}e^{-\phi} & & T^{\dot{\gamma}}_{\mathrm{C}}{}_{b}{}^{\alpha}_{\mathrm{A}} &=& \frac{i}{2}(\bar{\sigma}_b\sigma^{dc})^{\dot{\gamma}\alpha}F_{dc[\mathrm{CA}]}e^{-\phi} \\
T_{cb\mathrm{A}\alpha} &=& -(\epsilon\sigma_{cb})^{\gamma\beta}\Sigma_{(\gamma\beta\alpha)\mathrm{A}} & & T_{cb}{}^{\mathrm{A}\dot{\alpha}} &=& -(\epsilon\bar{\sigma}_{cb})_{\dot{\gamma}\dot{\beta}}\Sigma^{(\dot{\gamma}\dot{\beta}\dot{\alpha})\mathrm{A}} \\
\multicolumn{3}{l}{\quad -\frac{1}{4}\mathrm{tr}(\bar{\sigma}_{cb}\bar{\sigma}_{af})F^{af}{}_{[\mathrm{AF}]}\lambda^{\mathrm{F}}_{\alpha}} & & \multicolumn{3}{l}{\quad -\frac{1}{4}\mathrm{tr}(\sigma_{cb}\sigma_{af})F^{af[\mathrm{AF}]}\bar{\lambda}^{\dot{\alpha}}_{\mathrm{F}}} \\
\multicolumn{3}{l}{\quad -\frac{1}{12}\left(\delta^{af}_{cb} - \frac{i}{2}\varepsilon_{cb}{}^{af}\right)\bar{P}_a(\sigma_f\bar{\lambda}_{\mathrm{A}})_{\alpha}} & & \multicolumn{3}{l}{\quad -\frac{1}{12}\left(\delta^{af}_{cb} + \frac{i}{2}\varepsilon_{cb}{}^{af}\right)P_a(\bar{\sigma}_f\lambda^{\mathrm{A}})^{\alpha}}
\end{array}
\tag{F.2}
$$

with $U_a{}^{\mathrm{B}}{}_{\mathrm{A}} = -\frac{i}{8}(\lambda^{\mathrm{B}}\sigma_a\bar{\lambda}_{\mathrm{A}} - \frac{1}{2}\delta^{\mathrm{B}}_{\mathrm{A}}\lambda^{\mathrm{F}}\sigma_a\bar{\lambda}_{\mathrm{F}})$, P and $\bar{P}$ are given in equations (4.40) and (4.41), while $\Sigma_{(\gamma\beta\alpha)\mathrm{A}}$ and $\Sigma^{(\dot{\gamma}\dot{\beta}\dot{\alpha})\mathrm{A}}$ are the gravitino "Weyl" tensors.

Furthermore, the Lorentz curvature has components

$$\begin{aligned}
R_{\delta}{}^{\mathrm{D}}{}_{\gamma}{}^{\mathrm{C}}{}_{ba} &= 2\epsilon_{\delta\gamma}\mathrm{tr}(\sigma_{dc}\sigma_{ba})F^{dc[\mathrm{DC}]}e^{-\phi} & R^{\dot\delta}{}_{\mathrm{D}}{}^{\dot\gamma}{}_{\mathrm{C}ba} &= 2\epsilon^{\dot\delta\dot\gamma}\mathrm{tr}(\bar\sigma_{dc}\bar\sigma_{ba})F^{dc}{}_{[\mathrm{DC}]}e^{-\phi} \\
R^{\dot\delta}{}_{\mathrm{D}}{}_{\gamma}{}^{\mathrm{C}}{}_{ba} &= -4\epsilon_{dcba}U^{d\mathrm{D}}{}_{\mathrm{C}}(\sigma^{c}\varepsilon)_{\gamma}{}^{\dot\delta} & R_{\mathcal{C}\mathbf{u}\,ba} &= 0 \\
R_{\delta}{}^{\mathrm{D}}{}_{cba} &= -2i(\sigma_c)_{\delta\dot\alpha}(\epsilon\bar\sigma_{ba})_{\dot\gamma\dot\beta}\Sigma^{(\dot\gamma\dot\beta\dot\alpha)\mathrm{D}} & R^{\dot\delta}{}_{\mathrm{D}cba} &= -2i(\bar\sigma_c)^{\alpha\dot\delta}(\epsilon\sigma_{ba})^{\gamma\beta}\Sigma_{(\gamma\beta\alpha)\mathrm{D}} \\
&-\tfrac{i}{2}\mathrm{tr}(\sigma_{ba}\sigma_{ef})(\sigma_c\bar\lambda_{\mathrm{A}})_{\delta}F^{ef[\mathrm{DA}]}e^{-\phi} & &-\tfrac{i}{2}\mathrm{tr}(\bar\sigma_{ba}\bar\sigma_{ef})(\bar\sigma_c\lambda^{\mathrm{A}})^{\dot\delta}F^{ef}{}_{[\mathrm{DA}]}e^{-\phi} \\
&+\tfrac{i}{4}(\sigma_c\bar\sigma_e\sigma_{ba}\lambda^{\mathrm{D}})_{\delta}P^{e} & &+\tfrac{i}{4}(\bar\sigma_c\sigma_e\bar\sigma_{ba}\bar\lambda_{\mathrm{D}})^{\dot\delta}\bar P^{e}
\end{aligned} \tag{F.3}$$

and

$$\begin{aligned}
R_{dcba} &= (\epsilon\sigma_{dc})^{\delta\gamma}(\epsilon\sigma_{ba})^{\beta\alpha}V_{(\delta\gamma\beta\alpha)} + (\epsilon\bar\sigma_{dc})_{\dot\delta\dot\gamma}(\epsilon\bar\sigma_{ba})_{\dot\beta\dot\alpha}V^{(\dot\delta\dot\gamma\dot\beta\dot\alpha)} \\
&\quad +\frac{1}{2}\left(\eta_{db}R_{ca} - \eta_{da}R_{cb} + \eta_{ca}R_{db} - \eta_{cb}R_{da}\right) - \frac{1}{6}(\eta_{db}\eta_{ca} - \eta_{da}\eta_{cb})R
\end{aligned} \tag{F.4}$$

with the supercovariant Ricci tensor, $R_{db} = R_{dcba}\eta^{ca}$, given by

$$\begin{aligned}
R_{db} &= -2\mathcal{D}_d\phi\,\mathcal{D}_b\phi - \frac{1}{2}e^{-4\phi}H^*_d\,H^*_b - e^{-2\phi}F_{df[\mathrm{BA}]}F_b{}^{f[\mathrm{BA}]} + \frac{1}{4}\eta_{db}e^{-2\phi}F_{ef[\mathrm{BA}]}F^{ef[\mathrm{BA}]} \\
&\quad +\frac{1}{8}\sum_{db}\left\{i(\mathcal{D}_b\lambda^{\mathrm{F}})\sigma_d\bar\lambda_{\mathrm{F}} - i\lambda^{\mathrm{F}}\sigma_d\mathcal{D}_b\bar\lambda_{\mathrm{F}} + e^{-2\phi}H^*_d(\lambda^{\mathrm{F}}\sigma_b\bar\lambda_{\mathrm{F}})\right\} \\
&\quad -\frac{1}{32}(\lambda^{\mathrm{F}}\sigma_d\bar\lambda_{\mathrm{F}})\,(\lambda^{\mathrm{A}}\sigma_b\bar\lambda_{\mathrm{A}}) - \frac{1}{16}\eta_{db}(\lambda^{\mathrm{A}}\lambda^{\mathrm{F}})(\bar\lambda_{\mathrm{A}}\bar\lambda_{\mathrm{F}})
\end{aligned} \tag{F.5}$$

and the corresponding Ricci scalar, $R = R_{db}\eta^{db}$, which is then

$$R = -2\mathcal{D}^a\phi\mathcal{D}_a\phi - \frac{1}{2}H^{*a}H^{*}{}_{a}e^{-4\phi} + \frac{3}{4}e^{-2\phi}H^{*a}(\lambda^{\mathrm{A}}\sigma_a\bar\lambda_{\mathrm{A}}) + \frac{3}{8}(\lambda^{\mathrm{B}}\lambda^{\mathrm{A}})(\bar\lambda_{\mathrm{B}}\bar\lambda_{\mathrm{A}}). \tag{F.6}$$

The tensors $V_{(\delta\gamma\beta\alpha)}$ and $V^{(\dot\delta\dot\gamma\dot\beta\dot\alpha)}$ are components of the usual Weyl tensor. Like the gravitino Weyl tensors, $\Sigma_{(\gamma\beta\alpha)\mathrm{A}}$ and $\Sigma^{(\dot\gamma\dot\beta\dot\alpha)\mathrm{A}}$, their lowest components do not participate in the equations of motion.

As for the 2–form sector, besides the supercovariant field strength of the antisymmetric tensor, H_{cba}, the non-zero components of the 3–form H, which do not have central charge indices, are

$$H_{\mathrm{C}}{}^{\dot\gamma\mathrm{B}}{}_{\beta a} = -2i\delta_{\mathrm{C}}^{\mathrm{B}}(\sigma_a\epsilon)_{\beta}{}^{\dot\gamma}e^{2\phi} \qquad H_{\gamma}^{\mathrm{C}}{}_{ba} = 4(\sigma_{ba}\lambda^{\mathrm{C}})_{\gamma}e^{2\phi} \qquad H^{\dot\gamma}_{\mathrm{C}ba} = 4(\bar\sigma_{ba}\bar\lambda_{\mathrm{C}})^{\dot\gamma}e^{2\phi}\ . \tag{F.7}$$

The components with at least one central charge index, are related to the torsion components by

$$H_{\mathcal{D}\mathcal{C}\mathbf{u}} = T_{\mathcal{D}\mathcal{C}}{}^{\mathbf{z}}\mathfrak{g}_{\mathbf{zu}}, \tag{F.8}$$

with the metric $\mathfrak{g}_{\mathbf{zu}}$ defined in (4.34).

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